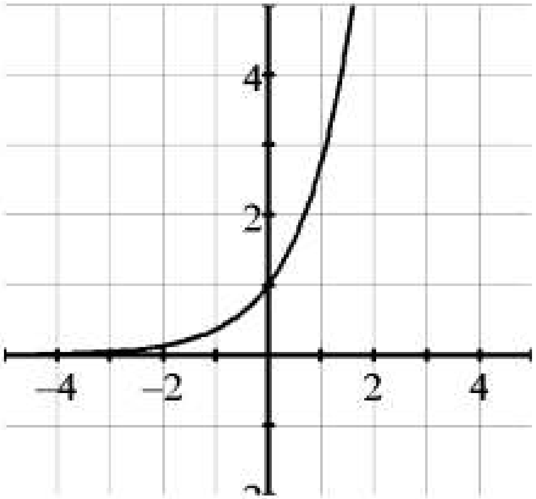
Taylor Polynomials and Taylor Series

What if our functions do not fit the Geometric Sum model? Can we fit a polynomial to other functions like we did for Power Series? Let's see!

How about $f(x)=e^x$. Our goal is to find a series that will mimic this function.



constant function:

linear function:

quadratic function:

cubic function:

In general terms:

Now let's see if we can find what all of those coefficients equal.

$f(x) =$

$f(0) =$

$f'(x) =$

$f'(0) =$

$f''(x) =$

$f''(0) =$

$f^{(n)}(x) =$

$f^{(n)}(0) =$

So. . . the polynomial that approximates a non-polynomial function (centered at 0) is called the ***nth degree Maclaurin polynomial for f.***

$$\boxed{P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots \frac{f^{(n)}(0)}{n!}x^n}$$

If we shift the center of our function (horizontal shift left or right), then we have a new center, *c*. This polynomial is called the ***nth degree Taylor polynomial for f at c***.

$$\boxed{P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots \frac{f^{(n)}(c)}{n!}(x-c)^n}$$

Find the Maclaurin polynomial of degree $n = 6$ for $f(x) = \cos x$. Then use it to approximate the value of $\cos(0.1)$.

$f(x) = \cos x$	at $x=0$

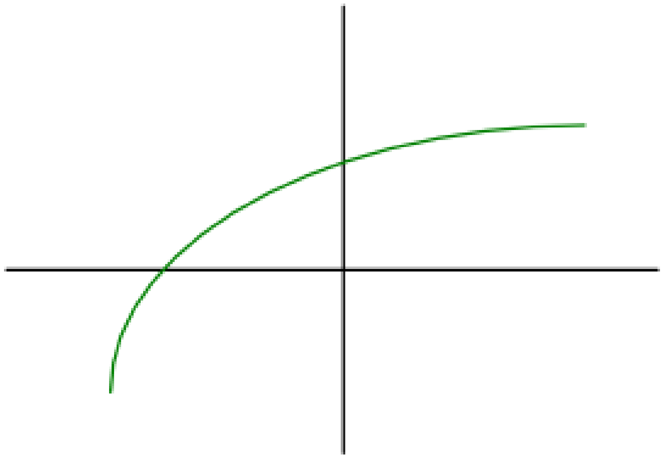
Find the Taylor polynomial of degree $n = 6$ for $f(x) = \ln x$ at $c=1$. Then use it to approximate the value of $ln(1.1)$.

$f(x) = \ln x$	at $x=1$

Use the Taylor approximation $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ for x near 0 to find:

$$\lim_{x\to 0} \frac{e^x - 1}{2x}$$

Given that $P_2(x) = a + bx + cx^2$ is the second-degree Taylor polynomial for f about $x=0$. What can you say about the signs of a, b and c if f has the graph pictured below? Explain your reasoning.



The more terms in a Taylor Polynomial, the better the approximation. What if we had an infinite number of terms? That would be a Taylor Series! (Aren't you excited??)

Taylor Series centered at x = c:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

If the series is centered at x = 0, it is called a Maclaurin series.

Let's make an important distinction before we move on:

A polynomial ends so there is no . . .

A series never ends so there is a . . .

Forget that and you lose points!!

Find a Taylor series for $f(x) = e^{5x}$ centered at c = 2. Give the first four nonzero terms and the general term.

There are four series that you must commit to memory for manipulation purposes (it will save you a LOT of time!)

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$ converges for: ________________	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for: ________________
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ converges for: ________________	$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converges for: ________________

We can manipulate Maclaurin series like we did power series, because they are centered at x = 0. We cannot manipulate any series that is not centered at 0. For that, we must find each derivative, evaluate it and write our series (yes - the long way!)

Use manipulation techniques to find a Maclaurin series for the following function.
Find the first four nonzero terms and the general term.

$$f(x) = \sin\left(x^2\right)$$

Use manipulation techniques to find a Maclaurin series for the following function.
Find the first four nonzero terms and the general term.

$$g(x) = x\cos(x)$$

Use manipulation techniques to find a Maclaurin series for the following function.
Find the first four nonzero terms and the general term.

$$h(x) = \frac{e^x + e^{-x}}{2}$$