Chapter 9 Right Triangles and Trigonometry

Addressed or Prepped VA SOL:

G.7 The student, given information in the form of a figure or statement, will prove two triangles are similar.

G.8 The student will solve problems, including practical problems, involving right triangles. This will include applying
a) the Pythagorean Theorem and its converse;
b) properties of special right triangles; and
c) trigonometric ratios.

SOL Progression

Middle School:
- Understand ratios and describe ratio relationships
- Decide whether two quantities are proportional (ratio tables, graphs)
- Represent proportional relationships with equations
- Identify corresponding sides and corresponding congruent angles of similar quadrilaterals and triangles

Algebra I:
- Solve linear equations in one variable
- Use linear equations to solve real-life problems
- Find the slope of a line
- Identify and use parallel and perpendicular lines in real-life problems

Geometry:
- Use the AA, SSS and SAS Similarity Theorems to prove triangles are similar
- Decide whether polygons are similar
- Use similarity criteria to solve problems about lengths, perimeters, and areas
- Prove the slope criteria using similar triangles
- Use the Triangle Proportionality Theorem and other proportionality theorems

Chapter 9 Right Triangles and Trigonometry

Section 9-1: The Pythagorean Theorem

SOL: G.8.a

Objective:

Use the Pythagorean Theorem
Use the Converse of the Pythagorean Theorem
Classify triangles

Vocabulary:

Hypotenuse – side in a right triangle opposite the right angle; largest side
Legs of a right triangle – the sides of the right triangle; the two smaller sides in triangle
Pythagorean triple – a set of three whole numbers (no fractions of decimals) that satisfy the Pythagorean Theorem

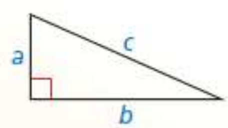
Core Concept:

Theorem

Theorem 9.1 Pythagorean Theorem

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

Proof Explorations 1 and 2, p. 423; Ex. 39, p. 444



$c^2 = a^2 + b^2$

Core Concept

Common Pythagorean Triples and Some of Their Multiples

3, 4, 5	**5, 12, 13**	**8, 15, 17**	**7, 24, 25**
6, 8, 10	10, 24, 26	16, 30, 34	14, 48, 50
9, 12, 15	15, 36, 39	24, 45, 51	21, 72, 75
$3x, 4x, 5x$	$5x, 12x, 13x$	$8x, 15x, 17x$	$7x, 24x, 25x$

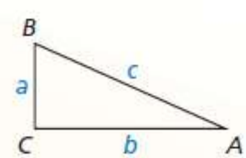
The most common Pythagorean triples are in bold. The other triples are the result of multiplying each integer in a bold-faced triple by the same factor.

Theorem

Theorem 9.2 Converse of the Pythagorean Theorem

If the square of the length of the longest side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

If $c^2 = a^2 + b^2$, then $\triangle ABC$ is a right triangle.



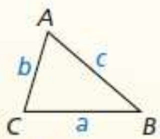
Proof Ex. 39, p. 430

Theorem

Theorem 9.3 Pythagorean Inequalities Theorem

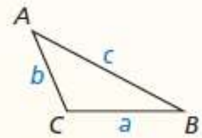
For any $\triangle ABC$, where c is the length of the longest side, the following statements are true.

If $c^2 < a^2 + b^2$, then $\triangle ABC$ is acute.



$c^2 < a^2 + b^2$

If $c^2 > a^2 + b^2$, then $\triangle ABC$ is obtuse.



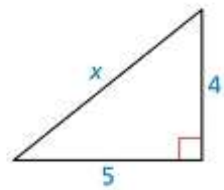
$c^2 > a^2 + b^2$

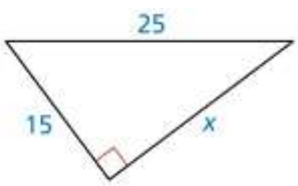
Proof Exs. 42 and 43, p. 430

Examples:

Example 1:

Find the value of x. Then tell whether the side lengths from a Pythagorean triple.



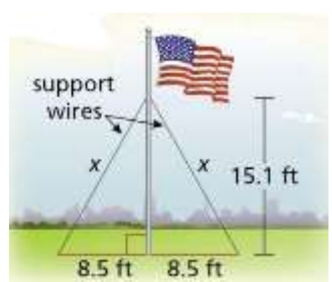


Example 2:

Find the value of x. Then tell whether the side lengths from a Pythagorean triple.

Example 3:

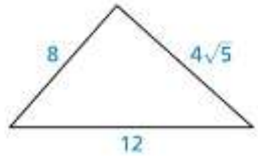
The flagpole show is supported by two wires. Use the Pythagorean Theorem to approximate the length of each wire.



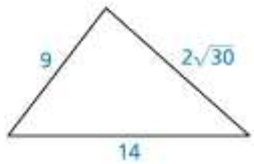
Example 4:

Tell whether each triangle is a right triangle.

a.



b.



Example 5:

Verify that segments with lengths of 14 meters, 15 meters and 11 meters form a triangle. Is the triangle *acute*, *right* or *obtuse*?

Concept Summary:

- The Pythagorean Theorem can be used to find the measures of the sides of a right triangle
- If the square of the largest side equals the sum of the squares of the smaller sides, then it's a right triangle
- Pythagorean Triples are all whole numbers
- Classify a triangle's largest angle by relationship of its largest side to the smaller two sides
    - If $c^2 < a^2 + b^2$, then an acute triangle
    - If $c^2 > a^2 + b^2$, then an obtuse triangle

Khan Academy Videos:

1. Introduction to the Pythagorean theorem
2. Pythagorean theorem part II
3. Pythagorean theorem with isosceles triangle

Homework: Pythagorean Worksheet

Reading: student notes section 9-2

Chapter 9 Right Triangles and Trigonometry

Section 9-2: Special Right Triangles

SOL: G.8.b

Objective:

Find side lengths in special right triangles
Solve real-life problems involving special right triangles

Vocabulary:

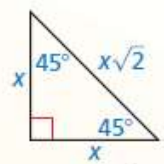
Right Isosceles triangle – a 45°-45°-90° triangle

Core Concept:

Theorem

Theorem 9.4 45°-45°-90° Triangle Theorem

In a 45°-45°-90° triangle, the hypotenuse is $\sqrt{2}$ times as long as each leg.



hypotenuse = leg • $\sqrt{2}$

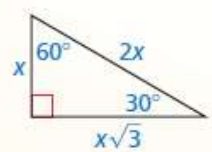
Proof Ex. 19, p. 436

Note: In a right isosceles triangle, either the legs have a $\sqrt{2}$ involved or the hypotenuse will have a $\sqrt{2}$ involved.

Theorem

Theorem 9.5 30°-60°-90° Triangle Theorem

In a 30°-60°-90° triangle, the hypotenuse is twice as long as the shorter leg, and the longer leg is $\sqrt{3}$ times as long as the shorter leg.



hypotenuse = shorter leg • 2
longer leg = shorter leg • $\sqrt{3}$

Proof Ex. 21, p. 436

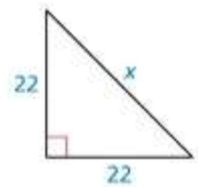
Note: some people teach the short leg is $\frac{1}{2} hypotenuse$ and the long leg is $\frac{1}{2} hypotenuse\sqrt{3}$
30-60-90 is a right scalene triangle

Examples:

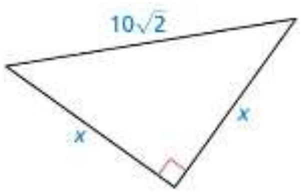
Example 1:

Find the value of *x*. Write your answer in simplest form.

a.

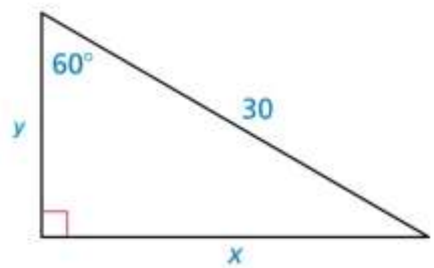


b.



Example 2:

Find the values of *x* and *y*. Write your answer in simplest form.

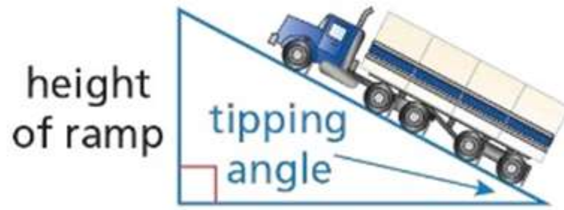


Example 3:

A warning sticker is shaped like an equilateral triangle with side length of 4 inches. Estimate the area of the sticker by finding the area of the equilateral triangle to the nearest tenth of an inch.

Example 4:

How high is the end of a 54-foot ramp when the tipping angle is 30°?



Concept Summary:

- Sometimes special case right triangles can be solved using Pythagorean theorem
- Sides opposite special angles summarized in table below:

Angle	Side Opposite
30°	$\frac{1}{2}hypotenuse$
45°	$\frac{1}{2}hypotenuse\sqrt{2}$
60°	$\frac{1}{2}hypotenuse\sqrt{3}$

Khan Academy Videos:

1. Special right triangles introduction, part 1, 45-45-90
2. Special right triangles introduction, part 2, 30-60-90
3. 30-60-90 triangle example problem

Homework: Special Case Triangle Worksheet

Reading: student notes section 9-3

Chapter 9 Right Triangles and Trigonometry

Section 9-3: Similar Right Triangles

SOL: G.7

Objective:

Identify similar triangles
Solve real-life problems involving similar triangles
Use geometric means

Vocabulary:

Arithmetic Mean – between two numbers is their average (add the two numbers and divide by 2)

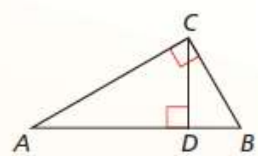
Geometric mean – a number, x, that satisfies $\frac{a}{x} = \frac{x}{b}$ or $x = \sqrt{ab}$
also the length of an altitude drawn to the hypotenuse of a right triangle

Core Concept:

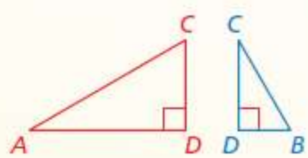
Theorem

Theorem 9.6 Right Triangle Similarity Theorem

If the altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other.



$\triangle CBD \sim \triangle ABC$, $\triangle ACD \sim \triangle ABC$, and $\triangle CBD \sim \triangle ACD$.



Proof Ex. 45, p. 444

Note: Two right triangles that share an angle are similar (via AA similarity Theorem)

Core Concept

Geometric Mean

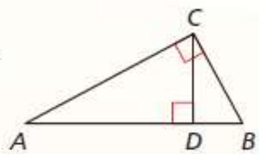
The **geometric mean** of two positive numbers a and b is the positive number x that satisfies $\frac{a}{x} = \frac{x}{b}$. So, $x^2 = ab$ and $x = \sqrt{ab}$.

Theorems

Theorem 9.7 Geometric Mean (Altitude) Theorem

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

The length of the altitude is the geometric mean of the lengths of the two segments of the hypotenuse.



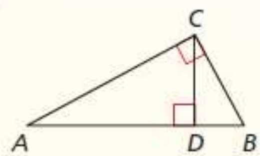
$CD^2 = AD \cdot BD$

Proof Ex. 41, p. 444

Theorem 9.8 Geometric Mean (Leg) Theorem

In a right triangle, the altitude from the right angle to the hypotenuse divides the hypotenuse into two segments.

The length of each leg of the right triangle is the geometric mean of the lengths of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.



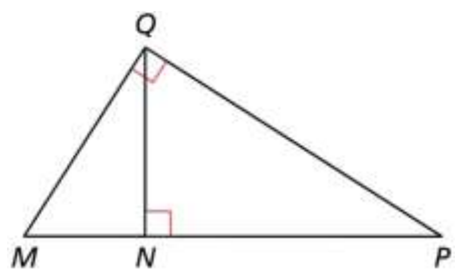
$CB^2 = DB \cdot AB$
$AC^2 = AD \cdot AB$

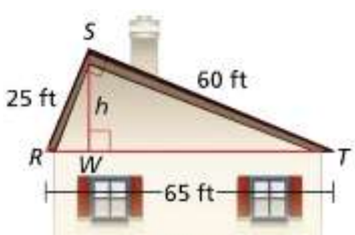
Proof Ex. 42, p. 444

Examples:

Example 1:

Identify the similar triangles in the diagram.





Example 2:

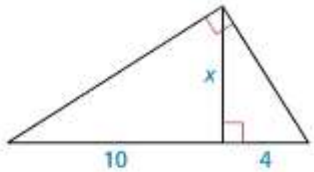
A roof has a cross section that is a right triangle. The diagram shows the approximate dimensions of this cross section. Find the height *h* of the roof.

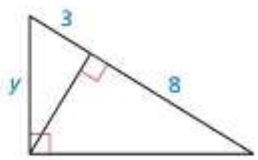
Example 3:

Find the geometric mean of 8 and 10

Example 4:

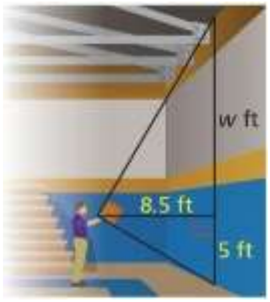
Find the value of each variable.

a. 

b. 

Example 5:

The vertical distance from the ground to your eye is 5.4 feet and the distance from you to the gym wall is 8.1 feet. Approximate the height of the gym wall.



Concept Summary:

- The arithmetic mean of two numbers is their average (add and divide by two)
- The geometric mean of two numbers is the square root of their product
- You can use the geometric mean to find the altitude of a right triangle

Khan Academy Videos:

1. Triangle similarity and the trigonometric ratios

Homework: "Means" Worksheet

Reading: student notes section 9-4

Section 9-4: The Tangent Ratio

SOL: G.8.c

Objective:

Use the tangent ratio
Solve real-life problems involving the tangent ratio

Vocabulary:

Angle of elevation – angle that an upward line of sight makes with a horizontal line
Tangent – ratio for acute angles that involves the lengths of the legs of a right triangle
Trigonometric ratio – ratio of the lengths of two sides in a right triangle

Core Concept:

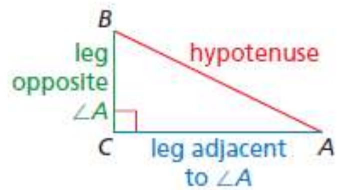
Core Concept

Tangent Ratio

Let $\triangle ABC$ be a right triangle with acute $\angle A$.

The tangent of $\angle A$ (written as $\tan A$) is defined as follows.

$$\tan A = \frac{\text{length of leg opposite } \angle A}{\text{length of leg adjacent to } \angle A} = \frac{BC}{AC}$$

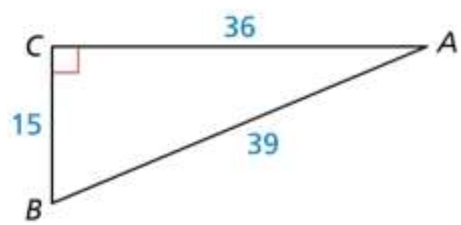


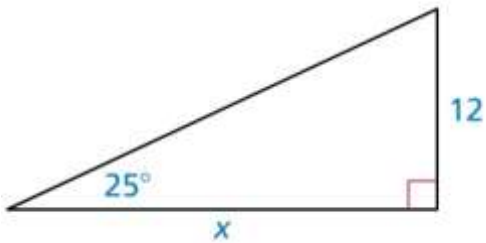
Note: You can Table of Trigonometric Ratios is available at BigIdeasMath.com to find decimal approximations of trigonometric ratios, if your phone does not have the proper calculator app.

Examples:

Example 1:

Find tan *A* and tan *B*. Write each answer as a fraction and as a decimal rounded to four places.





Example 2:

Find the value of *x*. Round your answer to the nearest tenth.

Example 3:

Use a special right triangle to find the tangent of a 60° angle.

Example 4:

You are measuring the height of a tree. You stand 40 feet from the base of the tree. The angle of elevation to the top of the tree is 65°. Find the height of the tree to the nearest foot.

Concept Summary:

- Tangent is used when we do not know the hypotenuse, but
  - we know either both legs (looking for an angle)
  - we know a leg and an angle (looking for the other leg)
- Used most often in finding the height of objects (knowing the distance from the object and the angle of elevation to the top of the object.

Khan Academy Videos: All trig videos together in next section

Homework: Trig Worksheet 1

Reading: student notes section 9-5

Chapter 9 Right Triangles and Trigonometry

Section 9-5: The Sine and Cosine Ratios

SOL: G.8.c

Objective:

Use the sine and cosine ratios
Find the sine and cosine of angle measures in special right triangles
Solve real-life problems involving sine and cosine ratios

Vocabulary:

Angle of depression – angle that a downward line of sight makes with a horizontal line
Cosine – trigonometric ratio that involves a leg (adjacent to the angle) and the hypotenuse
Sine – trigonometric ratio that involves a leg (opposite of the angle) and the hypotenuse

Core Concept:

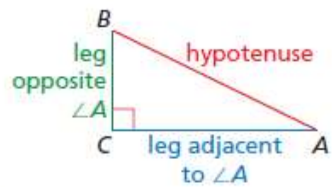
Core Concept

Sine and Cosine Ratios

Let $\triangle ABC$ be a right triangle with acute $\angle A$. The sine of $\angle A$ and cosine of $\angle A$ (written as $\sin A$ and $\cos A$) are defined as follows.

$$\sin A = \frac{\text{length of leg opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB}$$

$$\cos A = \frac{\text{length of leg adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB}$$



Core Concept

Sine and Cosine of Complementary Angles

The sine of an acute angle is equal to the cosine of its complement. The cosine of an acute angle is equal to the sine of its complement.

Let A and B be complementary angles. Then the following statements are true.

$\sin A = \cos(90° - A) = \cos B$ $\quad$ $\sin B = \cos(90° - B) = \cos A$

$\cos A = \sin(90° - A) = \sin B$ $\quad$ $\cos B = \sin(90° - B) = \sin A$

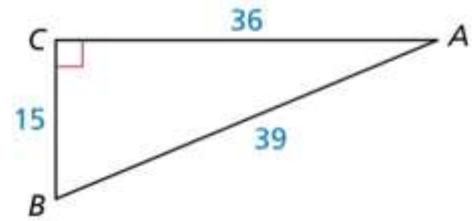
Note: Sin 45° = Cos 45° and from above concept Sin 30° = Cos 60° and Cos 30° = Sin 60°

Chapter 9 Right Triangles and Trigonometry

Examples:

Example 1:

Find sin *A*, sin *B*, cos *A*, cos *B*. Write each answer as a fraction and as a decimal rounded to four places.

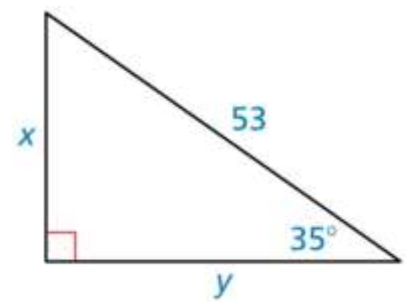


Example 2:

Write cos 69° in terms of sine.

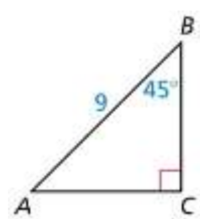
Example 3:

Find the values of *x* and *y* using sine and cosine. Round your answers to the nearest tenth.



Example 4:

Which ratios are equal to $\frac{\sqrt{2}}{2}$? Select all that apply

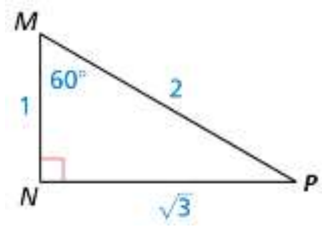


- Sin *A*
- Sin *B*
- Cos *A*
- Cos *B*
- Tan *A*
- Tan *B*

Example 5:

Which ratios are equal to $\frac{\sqrt{3}}{2}$? Select all that apply

- Sin *M*
- Sin *P*
- Cos *M*
- Cos *P*



Example 6:

You are skiing down a hill with an altitude of 800 feet. The angle of depression is 15°. Find the distance *x* you ski down the hill to the nearest foot.

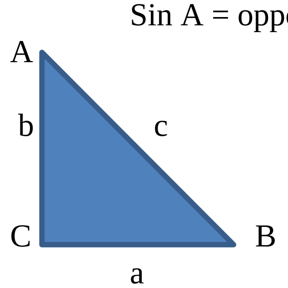
Concept Summary:

Sine and Cosine are the two only true trigonometric functions. All others are derived from these two (for example: Tan = Sin/Cos)

You must have or be looking for the hypotenuse

Due to the complementary nature of right triangles, Sin A = Cos B and Cos A = Sin B

Sin A = opposite/hypotenuse = a/c and Cos B = adjacent/hypotenuse = a/c



Khan Academy Videos:

1. Introduction to the trigonometric ratios
2. Trig ratios in right triangles
3. Solving for a side in right triangles with trig

Homework: Trig Worksheet 2

Note: problem 30 in homework has 20,000 feet as low altitude.

Reading: student notes section 9-5

Section 9-6: Solving Right Triangles

SOL: G.8

Objective:

Use inverse trigonometric ratios (used to find angular measure)
Solve right triangles

Vocabulary:

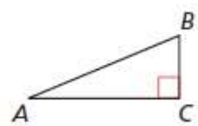
Inverse cosine – the measure of the angle whose cosine is the given ratio
Inverse sine – the measure of angle whose sine is the given ratio
Inverse tangent – the measure of angle whose tangent is the given ratio

Core Concept:

Core Concept

Inverse Trigonometric Ratios

Let $\angle A$ be an acute angle.



Inverse Tangent If $\tan A = x$, then $\tan^{-1} x = m\angle A$. $\quad \tan^{-1} \frac{BC}{AC} = m\angle A$

Inverse Sine If $\sin A = y$, then $\sin^{-1} y = m\angle A$. $\quad \sin^{-1} \frac{BC}{AB} = m\angle A$

Inverse Cosine If $\cos A = z$, then $\cos^{-1} z = m\angle A$. $\quad \cos^{-1} \frac{AC}{AB} = m\angle A$

Note: You can Table of Trigonometric Ratios is available at BigIdeasMath.com to find approximations of angles given trigonometric ratios, if your phone does not have the proper calculator app.

Core Concept

Solving a Right Triangle

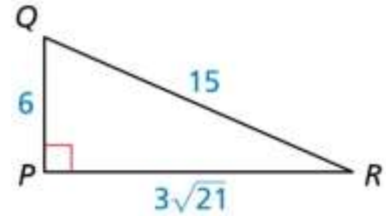
To **solve a right triangle** means to find all unknown side lengths and angle measures. You can solve a right triangle when you know either of the following.

- two side lengths
- one side length and the measure of one acute angle

Examples:

Example 1:

Determine which of the two acute angles has a sine of 0.4



Example 2:

Let $\angle A$, $\angle B$, and $\angle C$ be acute angles. Use a calculator to approximate the measures of $\angle A$, $\angle B$, and $\angle C$ to the nearest tenth of a degree.

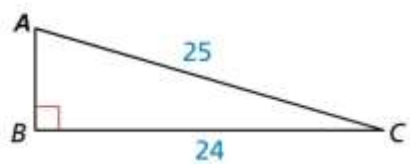
a. Tan $A = 3.29$

b. Sin $B = 0.55$

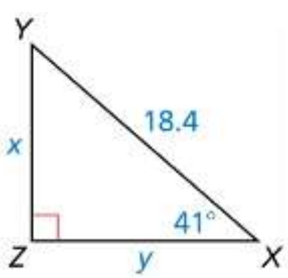
c. Cos $C = 0.87$

Example 3:

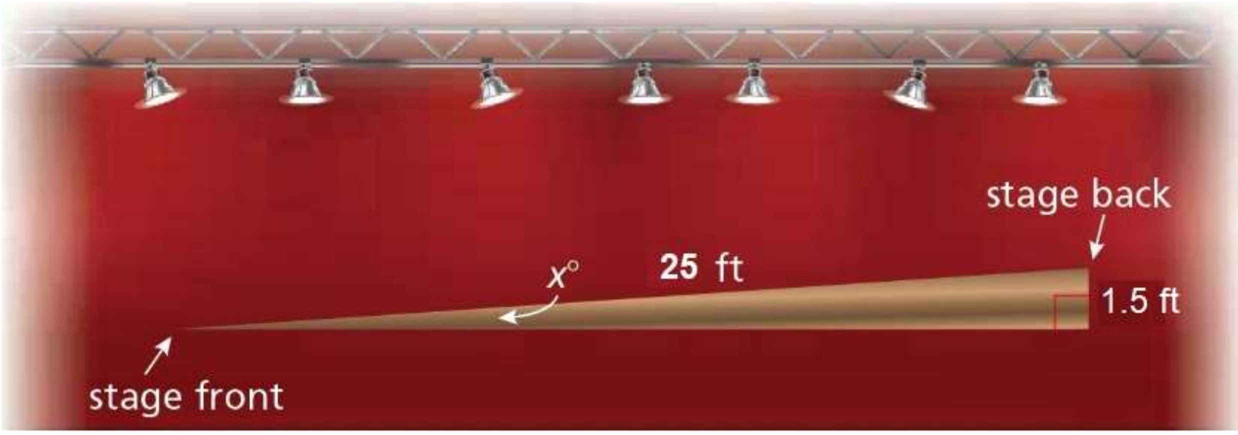
Solve the right triangle (find the missing sides and angles). Round decimal answers to the nearest tenth.



Example 4:



Solve the right triangle (find the missing sides and angles). Round decimal answers to the nearest tenth.



Example 5:

Another raked stage is 25 feet long from front to back with a total rise of 1.5 feet. You want the rake to be 5° or less. Is this raked stage within your desired range? Explain.

Concept Summary:

When you are looking for an angle, you use the inverse trig function
Calculators will display them as $sin^{-1}, cos^{-1}, tan^{-1}$

Khan Academy Videos: none related

Homework: Trig Worksheet 3 and Worksheet 4

Reading: student notes section 9-R

Chapter 9 Right Triangles and Trigonometry

Section 9-R: Trigonometry

SOL: G.8

Objective:
Use inverse trigonometric ratios
Solve right triangles

Vocabulary: none new

Concept Summary:

Pythagorean Theorem: $a^2 + b^2 = c^2$ where a and b are the legs and c is the hypotenuse (the longest side in a right triangle)

- Pythagorean Triples are all whole numbers (no decimals or fractions) that satisfy the theorem
- Classify a triangle's largest angle by relationship of its largest side, *c*, to the smaller two sides: *a* and *b*
  - If $c^2 < a^2 + b^2$, then an acute triangle
  - If $c^2 > a^2 + b^2$, then an obtuse triangle
- Used in right triangles (or to classify largest angle in a triangle)
- Basis of the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Special Case Right Triangles:

- Can always be solved using trig or and sometimes using Pythagorean Thrm
- Or the following table provides short cuts:

Angle	Side Opposite
30°	$\frac{1}{2} hypotenuse$
45°	$\frac{1}{2} hypotenuse\sqrt{2}$
60°	$\frac{1}{2} hypotenuse\sqrt{3}$

Geometric Means: $GM = \sqrt{ab}$

- Used to find the length of the altitude from the right angle to the hypotenuse
- Used in truss construction daily

Arithmetic Mean: $AM(average) = \frac{a+b}{2}$

- Analogous to midpoint
- Used in school a lot for grades

Trig Problems:

1. Label each side of the triangle as
   H for hypotenuse (opposite 90° and usually the diagonal side)
   A for side adjacent to given angle (A & H form the given angle)
   O for the side opposite the given angle
2. Determine using the information (sides and angles) given in the problem which of the trig functions you need to solve for variable

$$\sin(angle) = \frac{opp}{hyp} \qquad \cos(angle) = \frac{adj}{hyp} \qquad \tan(angle) = \frac{opp}{adj}$$

SOH CAH TOA

3. Set up an equation using the trig function and the variable
4. Solve for the variable (based on where the variable is)
   – Variable on top; multiply both sides by the bottom
   – Variable on bottom: variable and trig function trade places
   – Variable is angle: use inverse trig (2^{nd} key then trig key)

Khan Academy Videos:

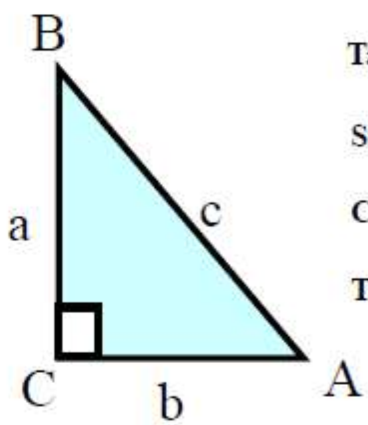
1. Right triangle word problem
2. Sine and cosine of complementary angles
3. Using complementary angles

Homework: Quiz Review Worksheet

Reading: student notes section 9-R

Trigonometry in Triangles

Trig Relationship between two acute angles



Sin A is opposite over hypotenuse: a/c

Cos A is adjacent over hypotenuse: b/c

Tan A is opposite over adjacent: a/b

Sin B is opposite over hypotenuse: b/c

Cos B is adjacent over hypotenuse: a/c

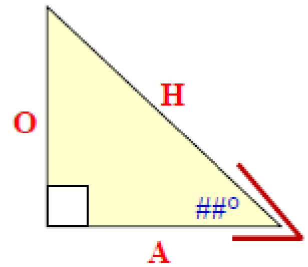
Tan B is opposite over adjacent: b/a

So, Sin A = Cos B and Cos A = Sin B

$a^2 + b^2 = c^2$ (from Pythagorean Theorem)

$m\angle A + m\angle B = 90°$ ($3\angle$'s of $\Delta = 180°$)

Trigonometry in Triangles



Steps for solving Trig problems

1. Mark perspective (given) angle

2. Label each side of the triangle as
   H for hypotenuse (opposite 90° and usually the diagonal side)
   A for side adjacent to given angle (A & H form the given angle)
   O for the side opposite the given angle

3. Determine using the information (sides and angles) given in the problem which of the trig functions you need to solve for variable

   sin (angle) = opp / hyp   cos (angle) = adj / hyp   tan(angle) = opp / adj

   SOH $S = \frac{O}{H}$ CAH $C = \frac{A}{H}$ TOA $T = \frac{O}{A}$

4. Set up an equation using the trig function and the variable

   $$\text{Trig function (angle°)} = \frac{\text{Some side}}{\text{Some other side}}$$

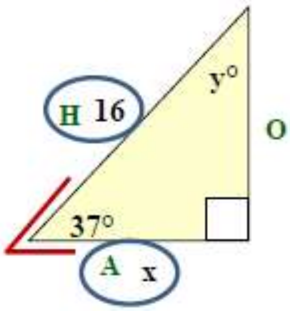
5. Solve for the variable (based on where the variable is)

   Variable on top; multiply both sides by the bottom
   Example 1 on next page

   Variable on bottom: variable and trig function trade places
   Example 2 on next page

   Variable is angle: use inverse trig (2nd key then trig key)
   Example 3 on next page

Example 1: (variable on top)



2) 16 is H, x is A and no value for O

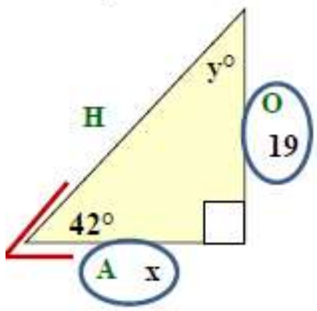
3) Since we have A and H we need to use cos

4) $\cos(37°) = \frac{x}{16}$ (x is on top multiply both sides by bottom)

5) $16\cos(37°) = x = 12.78$

Use 90 – 37 = 53 to find the other angle, y

Example 2: (variable on bottom)



2) 19 is O, x is A and no value for H

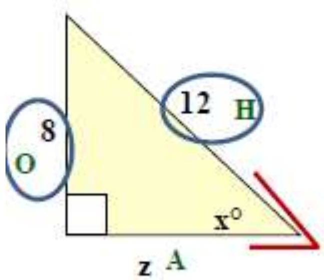
3) Since we have O and A (no H) we need to use tan

4) $\tan(42°) = \frac{19}{x}$ (x is on bottom then switch it with the trig function)

5) $x = \frac{19}{\tan(42°)} = 21.10$

Use 90 – 37 = 53 to find the other angle, y

Example 3: (variable is the angle)



2) 12 is H, 8 is O and no value for A -- x is the angle !

3) Since we have O and H we need to use sin

4) $\sin(x°) = \frac{8}{12}$ (x is angle use inverse sin)

5) $x = \sin^{-1}(8/12) = 48.19°$

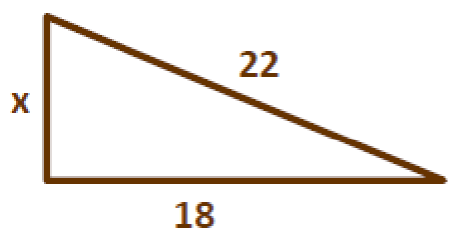
Use Pythagorean Theorem to find one missing side

$12^2 = z^2 + 8^2$ → $144 = z^2 + 64$ → $80 = z^2$ → $8.94 = z$

Pythagorean Theorem

On formula sheet: $a^2 + b^2 = c^2$ (c is the hypotenuse)
Variable in problem is not always the hypotenuse!!

Pythagorean Triple: 3 whole numbers that satisfy the formula
Common examples:
3, 4, 5   5, 12, 13   6, 8, 10   7, 24, 25   8, 15, 17



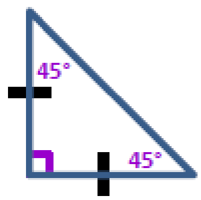
$$x^2 + 18^2 = 22^2$$
$$x^2 + 324 = 484$$
$$x^2 = 160$$
$$x \approx 12.65$$

Special Case Right Triangles

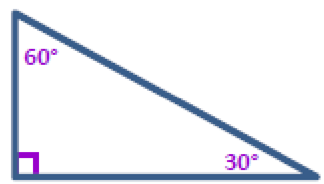
Sometimes can be solved using either Trig or Pythagorean Thrm

Two cases:

45 – 45 – 90
Right Isosceles



30 – 60 – 90
Right Scalene



Angle	Side Opposite	If hyp = 20, then
30	½ hypotenuse	½(20) = 10
45	½ hypotenuse × √2	½(20) × √2 = 14.14
60	½ hypotenuse × √3	½(20) × √3 = 17.32