4.5 Integration by Substitution

☺ What will you learn? ☺

- **Use pattern recognition to find an indefinite integral**
- **Use a change of variable to find an indefinite integral**
- **Use the General Power Rule for Integration to find an indefinite integral**
- **Use a change of variables to integrate a definite integral**
- **Evaluate a definite integral involving an even or odd function**

Pattern Recognition

Techniques for integrating composite functions

u -substitution

(comparable to the role of the Chain Rule)

2 Parts
- Pattern Recognition ⟶ perform substitution mentally
- Change of Variable ⟶ write substitution steps

Theorem 4.12 Antidifferentiation of a Composite Function

**Let *g* be a function whose range is an interval *I*,
and let *f* be a function this is continuous on *I*.
If g is differentiable on its domain
and *F* is an antiderivative of *f* on *I*,
then**

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

if $u = g(x)$, **then** $du = g'(x)\, dx$ **and**

$$\int f(u)\, du = F(u) + C$$

Exploration

Recognizing Patterns

The integrand in each of the following integrals fits the pattern of $f(g(x))\,g'(x)$.
Identify the pattern and use the result to evaluate the integral.

$$\int 2x\,(x^2 + 1)^4\,dx$$

$$\int 3x^2\,\sqrt{x^3 + 1}\,dx$$

$$\int \sec^2 x\,(\tan x + 3)\,dx$$

The next three integrals are similar to the first three.
Show how you can multiply and divide by a constant to evaluate these integrals.

$$\int x\,(x^2 + 1)^4\,dx$$

$$\int x^2\,\sqrt{x^3 + 1}\,dx$$

$$\int 2\sec^2 x\,(\tan x + 3)\,dx$$

Outside function

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Inside function

Derivative of Inside function

Example 1 - Recognizing the *f (g (x)) g'(x)* Pattern

Find

$$\int (x^2 + 1)^2 (2x)\,dx$$

Example 2 - Recognizing the $f(g(x))\,g'(x)$ Pattern

Find $$\int 5 \cos 5x \, dx$$

Example 3 - Multiplying & Dividing by a Constant

Find $$\int x\,(x^2 + 1)^2\,dx$$

Change of Variable

With a formal change of variables, you completely rewrite the integral in terms of ***u*** and ***du***
Useful for more complicated integrands

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du = F(u) + C$$

Example 4 - Change of Variables

Find

$$\int \sqrt{2x - 1}\, dx$$

Example 5 - Change of Variables

Find $\int x \sqrt{2x - 1}\, dx$

Example 6 - Change of Variables

Find

$$\int \sin^2 3x \cos 3x \, dx$$

Guidelines for Making a Change of Variables

1. **Choose a substitution** $u = g(x)$
   (Usually it is best to choose the INNER function)
2. **Compute** $du = g'(x)\,dx$
3. **Rewrite the integral in terms of the variable** u
4. **Find the resulting integral in terms of u**
5. **Replace u by g (x)**
6. **Check by differentiating**

The General Power Rule for Integration

one of the most common *u - subs*

Theorem 4.13 - The General Power Rule for Integration

If *g* is a differentiable function of *x*, then

$$\int [g(x)]^n g'(x)\,dx = \frac{[g(x)]^{n+1}}{n+1} + C$$

Equivalently, if *u* = *g (x)* , then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

Example 7 - Substitution and the General Power Rule

$$\int 3(3x-1)^4\,dx$$

$$\int (2x+1)(x^2+x)\,dx$$

$$\int 3x^2\sqrt{x^3-2}\,dx$$

$$\int \frac{-4x}{(1-2x^2)}\,dx$$

$$\int \cos^2 x \sin x\,dx$$

Change of Variables for Definite Integral

When using *u-sub* - it's often convenient to determine the limits of integration for the variable *u* rather than to convert the antiderivative back to *x* and evaluate at the original limits

Theorem 4.14 - Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a,b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 8 - Change of Variables

Evaluate

$$\int_0^1 x(x+1)^3 \, dx$$

Example 9 - Change of Variables

Evaluate

$$A = \int_1^5 \frac{x}{\sqrt{2x - 1}} \, dx$$

Integration of Even & Odd Functions

Occasionally, you can simplify the evaluation of a definite integral (over the interval that is symmetric about the y-axis or origin) by recognizing the integrand to be an even or odd function.

Theorem 4.15 - Integration of Even & Odd Functions

Let *f* be integrable on the closed interval *[-a,b]*

1. **If f is an even function, then** $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

2. **If f is an odd function, then** $\int_{-a}^{a} f(x)\,dx = 0$

Example 10 - Integration of an Odd Function

Evaluate $$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\, dx$$