

## 4.5 Integration by Substitution



**What will you learn?**



- Use pattern recognition to find an indefinite integral
- Use a change of variable to find an indefinite integral
- Use the General Power Rule for Integration to find an indefinite integral
- Use a change of variables to integrate a definite integral
- Evaluate a definite integral involving an even or odd function

## Pattern Recognition

Techniques for integrating composite functions

### u -substitution

(comparable to the role of the Chain Rule)

2 Parts  $\left\{ \begin{array}{l} \text{Pattern Recognition} \longrightarrow \text{perform substitution mentally} \\ \text{Change of Variable} \longrightarrow \text{write substitution steps} \end{array} \right.$

## Theorem 4.12 Antidifferentiation of a Composite Function

Let  $g$  be a function whose range is an interval  $I$ ,  
and let  $f$  be a function this is continuous on  $I$ .  
If  $g$  is differentiable on its domain  
and  $F$  is an antiderivative of  $f$  on  $I$ ,  
then

$$\int f(g(x)) g'(x) dx = F(g(x)) + C$$

if  $u = g(x)$ , then  $du = g'(x) dx$  and

$$\int f(u) du = F(u) + C$$

### Exploration

### Recognizing Patterns

The integrand in each of the following integrals fits the pattern of  $f(g(x))g'(x)$ . Identify the pattern and use the result to evaluate the integral.

$$\int 2x (x^2 + 1)^4 dx$$

$$\int 3x^2 \sqrt{x^3 + 1} dx$$

$$\int \sec^2 x (\tan x + 3) dx$$

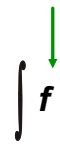
The next three integrals are similar to the first three. Show how you can multiply and divide by a constant to evaluate these integrals.

$$\int x (x^2 + 1)^4 dx$$

$$\int x^2 \sqrt{x^3 + 1} dx$$

$$\int 2 \sec^2 x (\tan x + 3) dx$$

Outside function



$$\int f(g(x)) g'(x) dx = F(g(x)) + C$$

Inside function



Derivative of  
Inside function



---

### Example 1 - Recognizing the $f(g(x)) g'(x)$ Pattern

Find

$$\int (x^2 + 1)^2 (2x) dx$$

**Example 2 - Recognizing the  $f(g(x))g'(x)$  Pattern**

Find

$$\int 5 \cos 5x \, dx$$

### Example 3 - Multiplying & Dividing by a Constant

Find

$$\int x (x^2 + 1)^2 dx$$

## Change of Variable

With a formal change of variables, you completely rewrite the integral in terms of  $u$  and  $du$   
Useful for more complicated integrands

$$\int f(g(x)) g'(x) dx = \int f(u) du = F(u) + C$$

---

### Example 4 - Change of Variables

Find

$$\int \sqrt{2x-1} dx$$

### Example 5 - Change of Variables

Find  $\int x \sqrt{2x - 1} \, dx$



### Example 6 - Change of Variables

Find

$$\int \sin^2 3x \cos 3x \, dx$$

### Guidelines for Making a Change of Variables

1. Choose a substitution  $u = g(x)$   
( Usually it is best to choose the INNER function)
2. Compute  $du = g'(x) dx$
3. Rewrite the integral in terms of the variable  $u$
4. Find the resulting integral in terms of  $u$
5. Replace  $u$  by  $g(x)$
6. Check by differentiating

## The General Power Rule for Integration

one of the most common  $u$  - subs

### Theorem 4.13 - The General Power Rule for Integration

If  $g$  is a differentiable function of  $x$ , then

$$\int [g(x)]^n g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + C$$

Equivalently, if  $u = g(x)$ , then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

### Example 7 - Substitution and the General Power Rule

$$\int 3 ( 3x - 1 ) ^ 4 dx$$

$$\int ( 2x + 1 ) ( x^2 + x ) dx$$

$$\int 3x^2 \sqrt{x^3 - 2} dx$$

$$\int \frac{-4x}{( 1 - 2x^2 )} dx$$

$$\int \cos^2 x \sin x dx$$

### Change of Variables for Definite Integral

When using *u-sub* - it's often convenient to determine the limits of integration for the variable  $u$  rather than to convert the antiderivative back to  $x$  and evaluate at the original limits

#### Theorem 4.14 - Change of Variables for Definite Integrals

If the function  $u = g(x)$  has a continuous derivative on the closed interval  $[a,b]$  and  $f$  is continuous on the range of  $g$ , then

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

### Example 8 - Change of Variables

Evaluate

$$\int_0^1 x (x + 1)^3 dx$$

### Example 9 - Change of Variables

Evaluate

$$A = \int_1^5 \frac{x}{\sqrt{2x-1}} dx$$

## Integration of Even & Odd Functions

Occasionally, you can simplify the evaluation of a definite integral (over the interval that is symmetric about the y-axis or origin) by recognizing the integrand to be an even or odd function.

### Theorem 4.15 - Integration of Even & Odd Functions

Let  $f$  be integrable on the closed interval  $[-a, b]$

1. If  $f$  is an even function, then 
$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

2. If  $f$  is an odd function, then 
$$\int_{-a}^a f(x) dx = 0$$



### Example 10 - Integration of an Odd Function

Evaluate  $\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x) dx$