5.5 Bases Other Than *e* and Applications

What will you learn?

- **Define exponential functions that have bases other than *e***
- **Differentiate and integrate exponential functions that have bases other than *e***
- **Use exponential functions to model compound interest and exponential growth**

Bases Other Than *e*

The base of the natural exponential function is *e*.
This "natural" base can be used to assign meaning to a general base *a*.

Definition of Exponential Function to Base *a*

If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)\, x}$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

These function follow the usual laws of exponents

$a^0 =$ $\quad$ $a^x a^y =$ $\quad$ $\dfrac{a^x}{a^y} =$ $\quad$ $(a^x)^y =$

Example 1- Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years.
A sample contains 1 gram of carbon-14.
How much will be present in 10,000 years?

Definition of Logarithmic Function to Base a

If *a* is a positive real numbers ($a \neq 1$) and *x* is any positive real number, then the logarithmic function to the base *a* is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x$$

Remember Properties of logs???

1. $\log_a 1 =$
2. $\log_a xy$
3. $\log_a x^n =$
4. $\log_a \frac{x}{y} =$

Properties of Inverse Functions

1. $y = a^x$ *iff* $x = \log_a y$
2. $a^{\log_a x} = x$, *for* $x > 0$
3. $\log_a a^x = x$, for all x

Example 2 - Bases Other Than e

Solve for x in each equation

a.) $3^x = \frac{1}{81}$

b.) $\log_2 x = -4$

Differentiation & Integration

You have 3 options for differentiation

1. Use definitions of $a^x = \log_a x$ and use rules for natural exponential & logarithmic functions
2. Use logarithmic differentiation
3. Use the following rules for bases other than *e*

Theorem 5.13 - Derivatives for Bases Other Than e

Let *a* be a positive real number ($a \neq 1$) and let *u* be a differentiable function of *x*

1. $\frac{d}{dx}[a^x] = (\ln a)\, a^x$
2. $\frac{d}{dx}[a^u] = (\ln a)\, a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)\, x}$
4. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)\, u} \frac{du}{dx}$

Example 3 - Differentiating Functions to Other Bases

Find the derivative of each function

a.) $y = 2^x$

b.) $y = 2^{3x}$

c.) $y = \log_{10} \cos x$

Integrating Exponential Functions to Other Bases

2 Options

1. Convert to base e, then integrate

2. Integrate directly using following formula

$$\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$$

Example 4 - Integrating an Exponential Function to Another Base

Find

$$\int 2^x \, dx$$

Theorem 5.14 - The Power Rule for Real Exponents

Let *a* be any real number and let *u* be a differentiable function of *x*

1. $\frac{d}{dx}[x^n] = n x^{n-1}$

2. $\frac{d}{dx}[u^n] = n u^{n-1} \frac{du}{dx}$

Example 5 - Comparing Variables and Constants

a.) $\frac{d}{dx}[e^e] =$

b.) $\frac{d}{dx}[e^x] =$

c.) $\frac{d}{dx}[x^e] =$

d.) $y = x^x$

Applications of Exponential Functions

Remember.....

$$A = P\left(1 + \frac{r}{n}\right)^{n}$$

Theorem 5.15 - A Limit Involving e

$$\lim_{x \to \infty}\left(1 + \frac{1}{x}\right)^{x} = \lim_{x \to \infty}\left(\frac{x+1}{x}\right)^{x} = e$$

Summary of Compounded Interest Formulas

Let **P = amount deposited**
t = number of years
A = Balance after t years
r = annual interest rate (decimal)
N = numbers of compoundings per year

Periodic Compounding

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuous Compounding

$$A = Pe^{rt}$$

Example 6 - Comparing Continuous and Quarterly Compounding

A deposit is made in an account that pays an annual interest rate of 5%.
Find the balance in the account at the end of 5 years if the interest is compounded

a.) quarterly

b.)monthly

c.) continuously

Example 7 - Bacterial Culture Growth

A bacterial culture is growing according to the logistic growth function

$$y = \frac{1.25}{1 + 0.25e^{-0.4t}} \qquad t \geq 0$$

where y is the weight of the culture in grams and t is the time in hours.
Find the weight of the culture after

a.) 0 hours

b.) 1 hour

c.) 10 hours

What is the limit as t approaches infinity?