3.4 Concavity & the Second Derivative

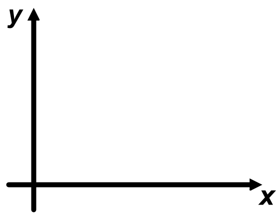
🙂 What will you learn? 🙂

- **Determine intervals on which a function is *concave upward* or *concave downward***
- **Find any *points of inflection***
- **Apply the second derivative test to find *relative extrema***

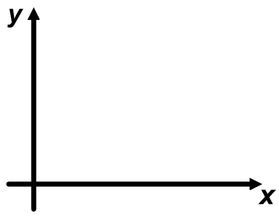
Definition of Concavity

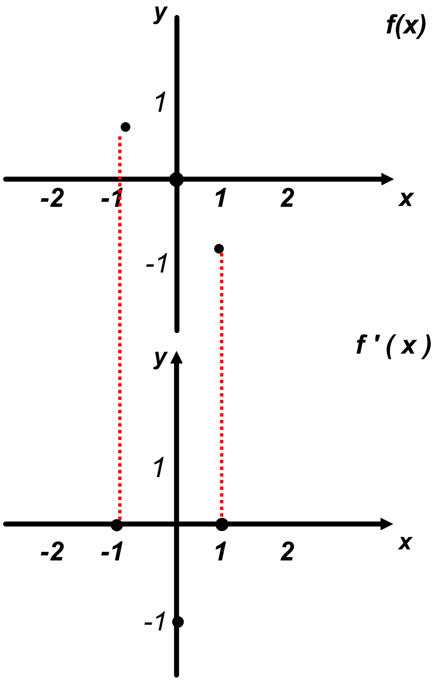
Let *f* be differentiable on an open interval *I*. The graph is concave upward on *I* if *f '* is increasing on the interval and concave downward on *I* if *f '* is decreasing on the interval.

1. *Let f be differentiable on an open interval I.*
   If the graph of f is concave upward on I,
   then the graph of f lies ABOVE all of its tangent lines



2. *Let f be differentiable on an open interval I.*
   If the graph of f is concave downward on I,
   then the graph of f lies BELOW all of its tangent lines





$f(x) = \frac{1}{3}x^3 - x$

$f'(x) = x^2 - 1$

To find the intervals on which the graph of a function *f* is concave upward or downward, you need to find the intervals on which *f '* is increasing or decreasing.

f (x)	***f ' (x)***
concave downward :	
concave upward :	

Theorem 3.7 - Test for Concavity

Let *f* be a function whose 2nd Derivative exists on an open interval *I*.

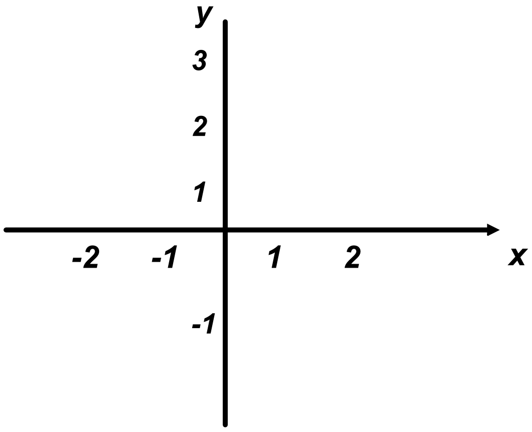
1. **If $f''(x) > 0$ for all *x* on *I*, then the graph is concave upward in *I*.**
2. **If $f''(x) < 0$ for all *x* on *I*, then the graph is concave downward in *I*.**

A third case would be if $f''(x) = 0$ for all *x* in *I*.
This would indicate that *f* is linear.
Concavity is not defined for a line.
A straight line ins neither concave up nor concave down.

Example 1 - Determining Concavity

Determine the open intervals on which the graph is concave upward or downward.

$$f(x) = \frac{6}{x^2 + 3}$$

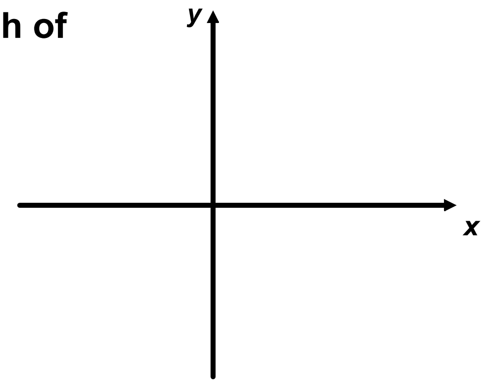


Example 2 - Determining Concavity

Determine the open interval on which the graph of

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

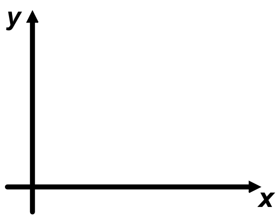
is concave upward or downward

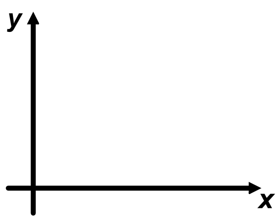


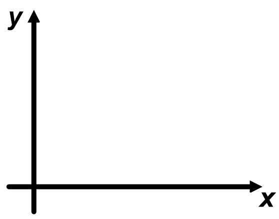
Point of Inflection

Let *f* be a function that is *continuous* on an open interval and let *c* be a point in the interval.

If the graph of *f* has a tangent line at this point *(c, f(c))*, then this point is a point of inflection of the graph of *f* if the concavity changes from upward to downward (or vice versa) at the point.







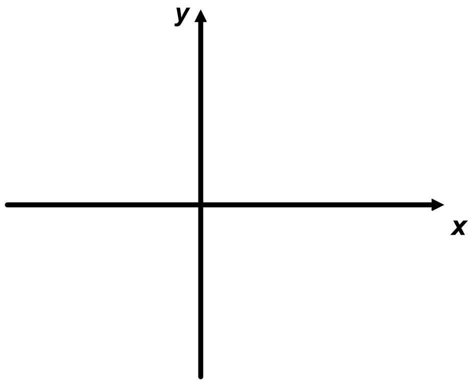
Theorem 3.8 - Points of Inflection

If *(c, f(c))* is a point of inflection of the graph of *f* , then either $f''(c) = 0$ or f'' *does not exist at* $x = c$.

Example 3 - Finding the Points of Inflection

Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3$$



The Second Derivative Test

In addition to testing for concavity - the 2nd derivative can be used to perform a simple test for the *relative maxima and minima*

Theorem - Second Derivative Test

Let f be a function s.t. $f'(c) = 0$ and f'' exists on the open interval containing c.

1. **If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$**
2. **If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$**

If $f''(c) = 0$, then the test fails.
That is f may have a relative maximum, minimum or neither.
In such case, you use the 1st Derivative Test.

Example 4 - Using the 2nd Derivative Test

Find the relative extrema for

$$f(x) = -3x^5 + 5x^3$$