3.9 Differentials

What will you learn?

- **Understand the concept of tangent line approximation**
- **Compare the value of the differential *dy*, with the actual change in *y*, Δy**
- **Estimate the propagated error using a differential**
- **Find the differential of a function using the differentiation formula**

Tangent Line Approximation

(Linear Approximation of *f* at *c*)

Using a tangent line to a graph to approximate the graph

Consider a function *f* that is differentiable at *c*

The equation of the tangent line a the point *(c, f(c))* is given by

$$y - f(c) = f'(c)(x - c) \qquad y_2 - y_1 = m(x_2 - x_1)$$

$$y = f(c) + f'(c)(x - c) \qquad \text{Solve for y}$$

- Because *c* is a constant, *y* is a linear function of *x*
- By restricting the values of *x* to be sufficiently close to *c*, the values of *y* can be used as approximations (to any desired accuracy) of the values of the functions *f*
- As $x \rightarrow c$, the limit of *y* is *f(c)*

Example 1 - Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point $(0, 1)$.
Then use the table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x)$							
$y = 1 + x$							

Differentials

$$y = f(c) + f'(c)\,\underbrace{(x - c)}_{\Delta x}$$

Tangent line at *(c, f(c))*

When Δx is small, Δy can be approximated as shown below

$$\Delta y = f(c + \Delta x) - f(c)$$

Actual change in y

$$\approx f'(c)\,\Delta x$$

Approximate change in y

$\Delta x \longrightarrow dx \longrightarrow$ ***the differential of x***

$f'(x)\,dx \longrightarrow dy \longrightarrow$ ***the differential of y***

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x.

The differential of x (denoted by dx) is any nonzero real number.

The differential of y (denoted by dy) is

$$dy = f'(x)\,dx$$

Change in y

$\Delta y \approx dy$

$\Delta y \approx f'(x)\,dx$

Example 2 - Comparing Δy and Δx

Let $y = x^2$

Find *dy* when $x = 1$ and $dx = .01$

Compare this value with Δy for $x = 1$ and $\Delta x = .01$

Error Propagation

Physicists and engineers tent to make liberal use of the approximation of Δy by dy

Let x represent the *measured value* of a variable
Let $x + \Delta x$ represent the *exact value*

Then Δx is the *error in measurement*

If the measured value is used to compute another value f (x),
then the difference between $f(x + \Delta x)$ and $f(x)$ is the propagated error.

Measurement Error

Propagated Error

$$f(\underbrace{x + \overbrace{\Delta x}^{\text{Measurement Error}}}_{\text{Exact Value}}) - f(\underbrace{x}_{\text{Measured Value}}) = \overbrace{\Delta y}^{\text{Propagated Error}}$$

Exact Value

Measured Value

Example 3 - Estimation of Error

The radius of a ball bearing is measured to be 0.7 inch.
If the measurement is correct within 0.01 inch, estimate the propagated error in the volume V of the ball bearing

Calculating Differentials

Each of the differentiation rules can be written in *differential form*

$$du = u'\,dx$$
$$dv = v'\,dx$$

Constant Multiple	$d[\,cu\,] = c\,du$
Sum or Difference	$d[\,u \pm v\,] = du \pm dv$
Product	$d[\,uv\,] = u\,dv + v\,du$
Quotient	$d\left[\,\frac{u}{v}\,\right] = \frac{v\,du - u\,dv}{v^2}$

Example 4 - Finding Differentials

Function	Derivative	Differential
a) $y = x^2$		
b) $y = 2 \sin x$		
c) $y = x \cos x$		
d) $y = 1/x$		

Example 5 - Finding the Differential of a Composite Function

$$y = f(x) = \sin 3x$$

Example 6 - Finding the Differential of a Composite Function

$$y = f(x) = (x^2 + 1)^{1/2}$$

Example 7 - Approximating Function Values

Use differentials to approximate $\sqrt{16.5}$