1.3 Evaluating Limits Analytically

☺ **What will you learn??** ☺

- **Evaluate a limit using properties of limits**
- **Develop and Use a Strategy for finding limits.**
- **Evaluate a limit using dividing out and rationalizing techniques.**
- **Evaluate a limit using the Squeeze Theorem.**

Properties of Limits

You have already learned that the limit of *f(x)* as x approaches *c* does not depend on the value of *f* at *x* = *c*.

It may be the case that the limit is *precisely f(c).*
In this case you may use **DIRECT SUBSTITUTION** to evaluate the limit

$$\lim_{x \to c} f(x) = f(c)$$

Substitute c for x

f(x) is **Continuous at c**

Theorem 1.1 - Some Basic Limits

Let *b* and *c* be real numbers and let *n* be a positive integer

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$

Example 1 - Evaluating Basic Limits

a. $\lim_{x \to 2} 3 =$ ______

b. $\lim_{x \to -4} x =$ ______

c. $\lim_{x \to 2} x^2 =$ __________

Theorem 1.2 Properties of Limits

Let b and c be real numbers,
let n be a positive integer, and
let f and g be functions with the following limits

$$\lim_{x \to c} f(x) = L \qquad \lim_{x \to c} g(x) = K$$

1. Scalar Multiple $\quad \lim_{x \to c} [\, b\, f(x) \,] = bL$

2. Sum or Difference $\quad \lim_{x \to c} [\, f(x) \pm g(x) \,] = L \pm K$

3. Product $\quad \lim_{x \to c} [\, f(x)g(x) \,] = LK$

4. Quotient $\quad \lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K} \qquad K \neq 0$

5. Power $\quad \lim_{x \to c} [\, f(x) \,]^n = L^n$

Example 2 - The Limit of a Polynomial

$$\lim_{x \to 2} (4x^2 + 3) =$$

Direct Substitution is valid for all polynomial and rational functions with a nonzero denominator

Theorem 1.3 - Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c)$$

If r is a rational function given by $r(x) = p(x) / q(x)$ and c is a real number s.t. $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$$

Example 3 - The Limit of a Rational Function

Find the limit:

$$\lim_{x \quad 1} \frac{x^2 + x + 2}{x + 1} =$$

Theorem 1.4 - The Limit of a Function Involving a Radical

Let n be a positive integer.
The following limit is valid for all c if n is odd,
and is valid for $c > 0$ if n is even

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Theorem 1.5 - The Limit of a Composite Function

If f and g are functions s.t.

$$\lim_{x \to c} g(x) = L \qquad \lim_{x \to L} f(x) = f(L)$$

then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

Example 4 -The Limit of a Composite Function

a.) Because $\lim_{x \to 0} (x^2 + 4) =$ ________ and $\lim_{x \to 4} \sqrt{x} =$ ________

it follows that $\lim_{x \to 0} \sqrt{x^2 + 4} =$ ________

b.) Because $\lim_{x \to 3} (2x^2 - 10) =$ ________ $\lim_{x \to 8} \sqrt[3]{x} =$ ________

it follows that $\lim_{x \to 3} \sqrt[3]{2x^2 - 10} =$ ________

Theorem 1.6 - Limits of Trig Functions

Let *c* be a real number in the domain of the given trig function.

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

Example 5 - Limits of Trig Functions

a.) $\lim_{x \to 0} \tan x =$

b.) $\lim_{x \to \pi} (x \cos x) =$

c.) $\lim_{x \to 0} \sin^2 x =$

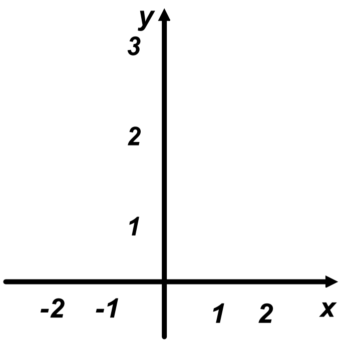
Theorem 1.7 - Finding the Limit of a Function

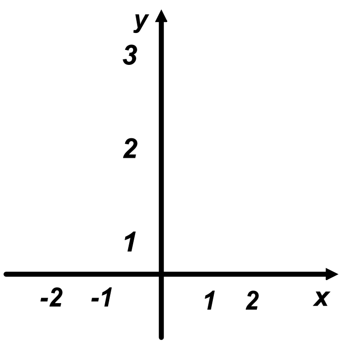
Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c.
If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Example 6 - Finding the Limit of a Function

Find the limit $\displaystyle \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$





A Strategy for Finding Limits

1. Learn to *recognize* which limits can be evaluated by direct substitution.

2. If the limit of *f(x)* as *x* approaches *c* cannot be evaluated by direct substitution, try to find a function *g* that agrees with *f* for all *x* other than *x* = *c*.

3. Apply Theorem 1.7 to conclude analytically

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c)$$

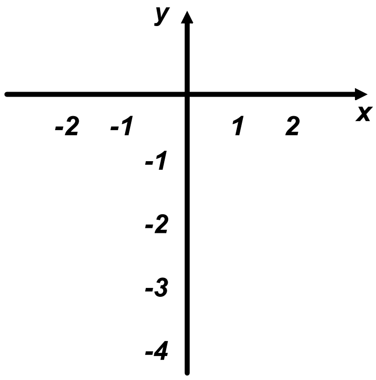
4. Use a graph of a table to reinforce your conclusion.

Dividing Out and Rationalizing Techniques

Example 7 - Dividing Out Technique

Find the Limit

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$



$$\frac{0}{0} \longrightarrow$$ **Indeterminate Form**

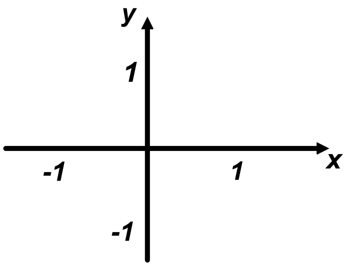
you cannot determine the limit from this form alone

If you encounter this form when you try to evaluate a limit - you must rewrite the fraction so that the new fraction does not have 0 as its limit

Example 8 - Rationalizing Technique

Find the limit

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$



x	-.25	-.1	-.01	-.001	0	.001	.01	.1	.25
f(x)									

Theorem 1.8 - The Squeeze Theorem

If $h(x) \le f(x)$ for all x on an open interval c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L

Theorem 1.9 - Two Special Trig Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Example 9 - A Limit Involving a Trig Function

Find the Limit

$$\lim_{x \to 0} \frac{\tan x}{x}$$

Example 10 - A Limit Involving a Trig Function

Find the Limit

$$\lim_{x \to 0} \frac{\sin 4x}{x}$$