11.2 Techniques of Evaluating Limits

Techniques for evaluating limits for which direct substitution fails

Dividing Out Technique

Example 1

Find the limit:

$$\lim_{x \to -3} \frac{x2 + x - 6}{x + 3}$$

Direct substitution fails ....Why?

See p. 760; exercise 7

If 2 functions agree at all but a single number *c*, they must have an identical limit behavior at $x = c$

The dividing out technique should be applied only when direct substitution produces 0 in the numerator and denominator.
The resulting fraction 0/0 has no meaning as a real number. It is called an *INDETERMINATE FORM* because you cannot, from the form alone, determine the limit.
When you try to evaluate a limit of a rational fraction by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor.
After factoring out and dividing out, you should try direct substitution again.

Example 2 - Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

See p. 760; exercise 9

Rationalizing Technique

Another way to find the limit of a function is to first rationalize the numerator of the function.

Rationalizing the numerator means multiplying the numerator and denominator by the *conjugate* of the numerator.

Example 3 - Rationalizing Technique

Find the limit:

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

See p. 760; exercise 11

Using Technology

Example 4 - Approximating a Limit

Approximate the limit: $\lim_{x \to 0} (1 + x)^{1/x}$

Numerical

Graphical

See p. 761; exercise 27

Example 5 - Approximating a Limit Graphically

Approximate the limit: $$\lim_{x \to 0} \frac{\sin x}{x}$$

See p. 761; exercise 31

One-Sided Limits

One way in which a limit can fail to exist is when a function approaches a different value from the left side of *c* than it approaches from the right side of *c*.

$$\lim_{x \to c^-} f(x) = L$$ **Limit from the LEFT**

$$\lim_{x \to c^+} f(x) = L$$ **Limit from the RIGHT**

Example 6 - Evaluating One-Sided Limits

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for:

$$f(x) = \frac{|2x|}{x}$$

See p. 761; exercise 43

Notice the limits from each side are different....therefore, the limit does not exist.

For a limit to exist as $x \to c$, it must be true that both one-sided limits *exist and are equal* .

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

iff both the left and right limits *exist and are equal to L.*

Example 7 - Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

See p. 761; exercise 47

Example 8 - Comparing Limits from Left & Right

To ship a package overnight, a delivery service charges \$8 for the first pound and \$2 for each additional pound or portion of a pound.

Let x represent the weight of a package.
Let $f(x)$ represent the shipping cost.

Show that the limit of $f(x)$ as $x \to 2$ does NOT exist.

$$f(x) = \begin{cases} 8, & 0 < x \le 1 \\ 10, & 1 \le x \le 2 \\ 12, & 2 \le x \le 3 \end{cases}$$

A Limit from Caclulus

For the function $f(x) = x2 - 1$

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

See p. 761; exercise 63

Difference Quotient

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Direct substitiution into the difference quotient always produces the indeternminate form 0/ 0