Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, the limit of $f(x)$ as x approaches c is L.

$$\lim_{x \to c} f(x) = L$$

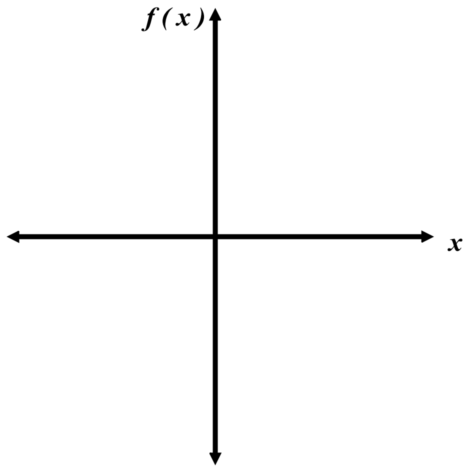
Example 2 - Estimating a Limit

Use a table to estimate numerically the limit:

$$\lim_{x \to 2} (3x - 2)$$

Approaching 2 from the left — *Approaching 2 from the right*

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$				?			



See p. 750; exercise 3

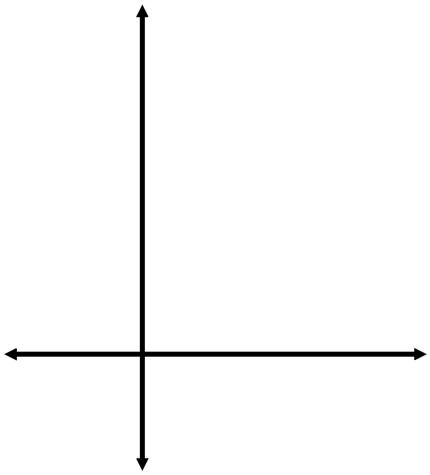
Example 3 - Estimating a Limit Numerically

Use a table to estimate numerically the limit:

$$\lim_{x \to 0} \frac{x}{\sqrt{x+1} - 1}$$

approaching 0 from left → *approaching 0 from right* ←

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$				?			



See p. 750; exercise 5

Note : $f(x)$ has a limit when $x \to 0$ even though the function is not defined at $x = 0$

The existence or nonexistence of f (x) at $x = c$ has NO bearing on the limit of $f(x)$ as $x \to c$

Example 4 - Using a Graphing Calculator to Find a Limit

Estimate the limit: $$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$$

Numerical

Table Feature

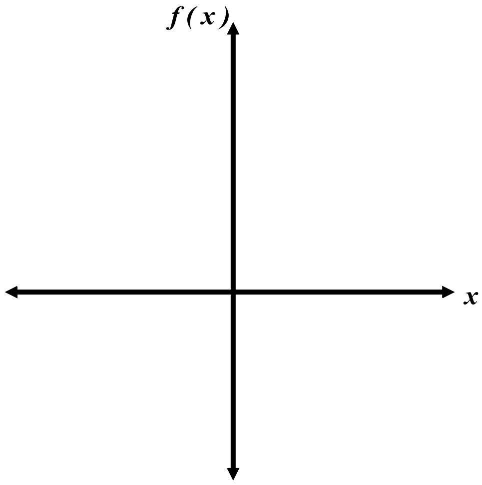
Graphical

See p. 750; exercise 7

Example 5 - Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where f is defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}$$



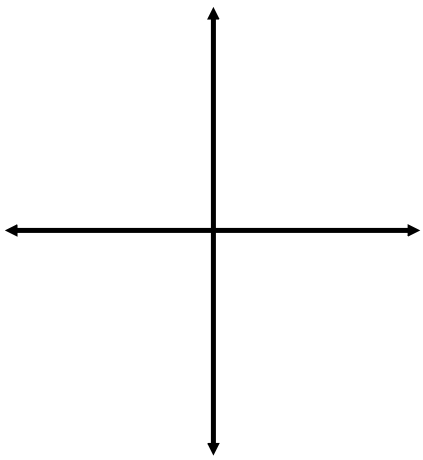
See . p 750; exercise 15

Limits that Fail to Exist

Example 6 - Comparing Left and Right Behavior

Show that the limit does not exist

$$\lim_{x \to 0} \frac{|x|}{x}$$

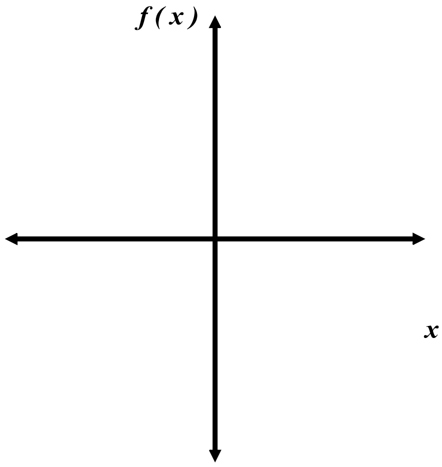


See p. 751; exercise 19

Example 7 - Unbounded Behavior

Discuss the existence of the limit:

$$\lim_{x \to 0} \frac{1}{x^2}$$



See p. 751; exercise 20

Example 8 - Oscillating Behavior

Discuss the existence of the limit:

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin\left(\frac{1}{x}\right)$							

See p. 751; exercise 21

Conditions Under Which Limits Do NOT Exist

The limit of $f(x)$ as $x \to c$ does not exist if any of the following conditions are true:

1. **$f(x)$ approaches a different number from the right side of c than it approaches from the left side of c**
2. **$f(x)$ increases or decreases without bound as $x \to c$**
3. **$f(x)$ oscillates between 2 fixed vlaues as $x \to c$**

Properties of Limits and Direct Substitution

Sometimes the limit of $f(x)$ as $x \to c$ is $f(c)$

Direct Substitution

$$\lim_{x \to c} f(x) = f(c)$$

Basic Limits

Let b and c be real numbers and let n be a positive integer,

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for n even and $c > 0$

Trig Functions can also be included in this list

ie) $\lim_{x \to \pi} \sin x = \sin \pi$
$= 0$

$\lim_{x \to 0} \cos x = \cos 0$
$= 1$

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. **Scalar Multiple:** $\lim_{x \to c} [b f(x)] = bL$
2. **Sum or Difference** $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. **Product :** $\lim_{x \to c} [f(x) g(x)] = LK$
4. **Quotient:** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ $\quad K \neq 0$
5. **Power:** $\lim_{x \to c} [f(x)]^n = L^n$

Let $\lim_{x \to 3} f(x) = 7$ and $\lim_{x \to 3} g(x) = 12$

a.) $\lim_{x \to 3} [f(x) - g(x)] = -5$

b.) $\lim_{x \to 3} [f(x) g(x)] = 84$

c.) $\lim_{x \to 3} [g(x)]^{1/2} = 2\sqrt{3}$

Example 9 - Direct Substitution & Properties of Limits

Find each limit.

a.) $\lim_{x \to 4} x^2$

b.) $\lim_{x \to 4} 5x$

c.) $\lim_{x \to 9} \sqrt{x}$

d.) $\lim_{x \to \pi} (x \cos x)$

Limits of Polynomial & Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c)$$

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number s.t. $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$$

Example 10 - Evaluating Limits by Direct Substitution

Find each limit.

a.) $\lim_{x \to -1} (x^2 + x - 6)$

b.) $\lim_{x \to -1} \frac{(x^2 + x - 6)}{x + 3}$

See p. 752; exercise 39