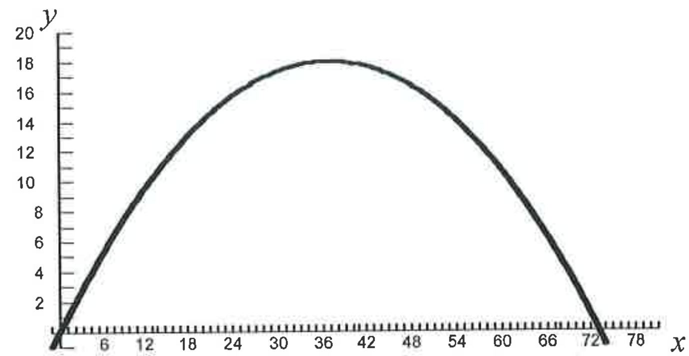
Key

Up until now, you have been representing a graph by a single equation involving two variables. Type #4
We will now introduce a third variable called a parameter into our equations.

Consider the path of an object that is propelled into the air at an angle of 45^0. If the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

$$y = -\frac{x^2}{72} + x \longrightarrow \text{Rectangular Equation}$$

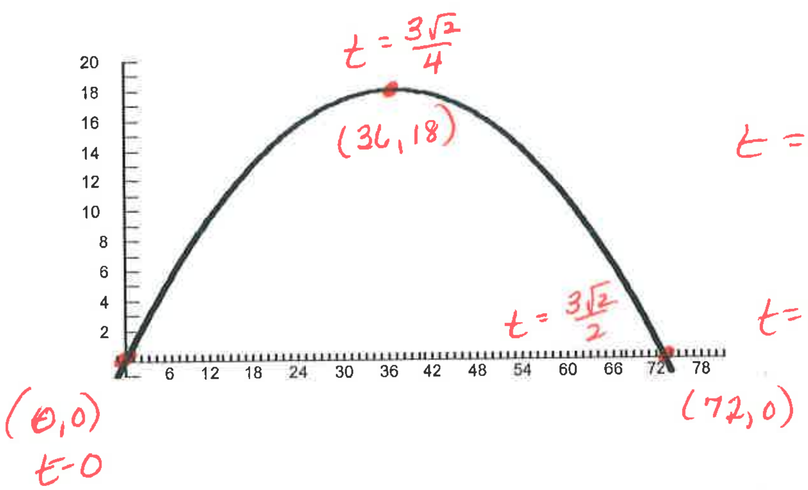


This equation does not tell the whole story! It tells *where* the object has been, but it does not tell *when* the object was at a given point (x, y) on the path.

To determine this time, you can introduce a third variable, *t*, called a **parameter**.
It is possible to write both *x* and *y* as functions of t to obtain **parametric equations**.

$$x = 24\sqrt{2}t \longrightarrow \text{Parametric equation for } x$$

$$y = -16t^2 + 24\sqrt{2}t \longrightarrow \text{Parametric equation for } y$$



$t = 0 \quad x = 0 \quad y = 0$
$(0, 0)$

$t = 1 \quad x = 24\sqrt{2} \quad y = 24\sqrt{2} - 16$

x and y are continuous function of t
path → PLANE CURVE

Definition of a Plane Curve

> If f and g are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a **plane curve** C. The equations given by:
>
> $$x = f(t)$$
> $$y = g(t)$$
>
> are **parametric equations** for C , and t is the **parameter**.

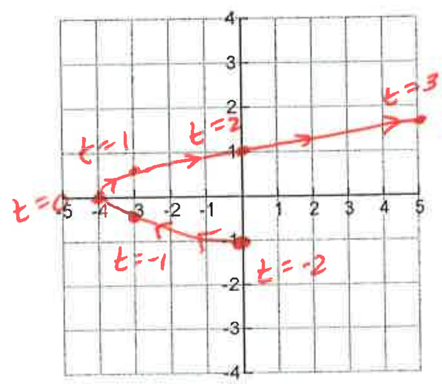
Sketching a Plane Curve

- Plot points in the xy-plane
- Each x and y is determined by a chosen value for t
- Plot the ordered pairs in order of *increasing* values of t – **orientation**

Example: Sketch the curve given by the parametric equations:

$$x = t^2 - 4 \qquad \boxed{-2 \le t \le 3}$$
$$y = \frac{t}{2}$$

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

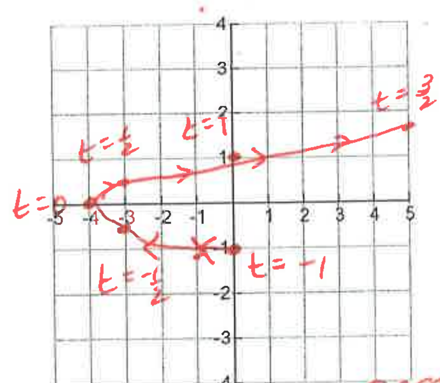


Two different sets of parametric equations can have the same graph

SAME AS ABOVE!

$$x = 4t^2 - 4 \qquad -1 \le t \le \frac{3}{2}$$
$$y = t$$

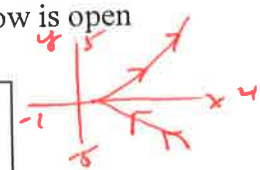
t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
x	0	3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$



OBJ IS MOVING FASTER

Using your Graphing Calculator in Parametric Mode

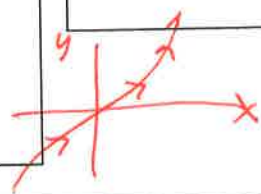
- Parametric Mode
- Window – set min and max for *x, y and t*
- Allows you to graph equations that are not functions
- Notice that orientation is shown when window is open

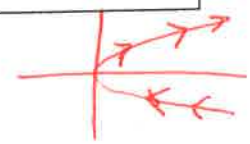


$x = t^2$ $\quad -4 \le T \le 4;$ *Step* 0.1
$y = t^3$ $\quad -1 \le x \le 4;$ *Scale* 1
$\quad -5 \le y \le 5;$ *Scale* 1

$x = t^2$ $\quad -4 \le T \le 4;$ *Step* 0.1
$y = t$ $\quad -1 \le x \le 4;$ *Scale* 1
$\quad -3 \le y \le 3;$ *Scale* 1

$x = t$ $\quad -4 \le T \le 4;$ *Step* 0.1
$y = t^3$ $\quad -5 \le x \le 5;$ *Scale* 1
$\quad -70 \le y \le 70;$ *Scale* 10





Eliminating the Parameter

Parametric Equations ⇨ Solve for t in one equation ⇨ Substitute into 2^{nd} equation ⇨ Rectangular Equation

$x = t^2 - 4$ $\quad t = 2y$ $\quad x = (2y)^2 - 4$ $\quad x = 4y^2 - 4$

$y = \frac{1}{2}t$

$x = 4(y - 0)^2 - 4$ h k

opens →

Rectangular equation is a parabola with horizontal axis and vertex at (-4,0)

Example 1: Identify and graph, using orientation, the curve represented by the equations:

$x = \frac{1}{\sqrt{t+1}}$ $\quad t > -1 \quad \therefore x > 0$ $\quad y = \frac{t}{t+1}$

t	0	3
x	1	$\frac{1}{2}$
y	0	$\frac{3}{4}$

① Solve for t:

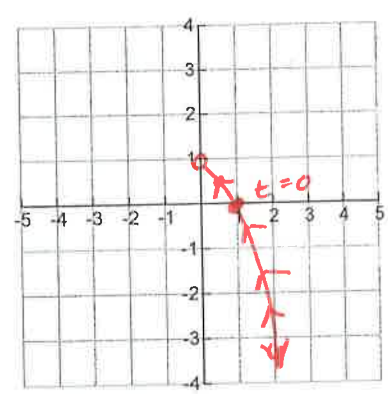
$x^2 = \frac{1}{t+1}$

$t + 1 = \frac{1}{x^2}$

$t = \frac{1}{x^2} - 1$

② Subst.

$\frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1} = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2}} = \boxed{1 - x^2 \quad x > 0}$



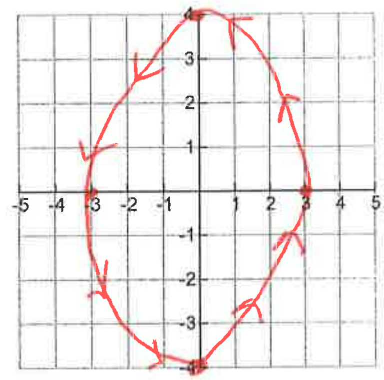
Example 2: Identify and graph, using orientation, the curve represented by the equations:

$x = 3\cos\theta$ D: $[0, 2\pi]$ R: $[-3, 3]$

$y = 4\sin\theta$ D: $[0, 2\pi]$ $[-4, 4]$

$0 \le \theta \le 2\pi$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	3	0	-3		
y	0	4	0		



$\cos\theta = \frac{x}{3}$ $\quad \sin\theta = \frac{y}{4}$

$\sin^2\theta + \cos^2\theta = 1$

$\left(\frac{y}{4}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$

$\frac{y^2}{16} + \frac{x^2}{9} = 1$

ELLIPSE COUNTERCLOCKWISE ORIENT.

C (0,0)

a = 4 vertical m.a.

b = 3

Example 3: Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the parameters:

a) $t = x$

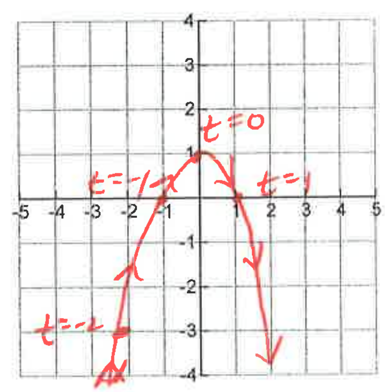
t	-2	-1	0	1
x	-2	-1	0	1
y	-3	0	1	0

D: R
R: R

$x = t$

$y = 1 - t^2$

D: R
R: $[-\infty, 1]$



b) $t = 1 - x$

D: R
R: R

t	-1	0	1	2
x	2	1	0	-1
y	-3	0	1	

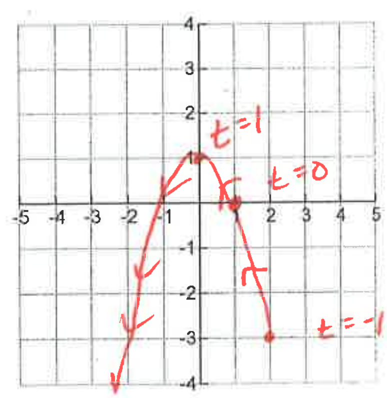
$x = 1 - t$

$y = 1 - (1-t)^2$

$1 - (1 - 2t + t^2)$

$2t - t^2$

D: R
R: $(-\infty, 1]$



Match the set of parametric equations with its graph.

1) $x = t$ $\quad$ D: ℝ, R: ℝ

$y = t + 2$ $\quad$ D: ℝ, R: ℝ

$\boxed{y = x + 2}$

$\boxed{c}$

2) $x = t^2$ $\quad$ D: ℝ, R: $x \geq 0$

$y = t - 2$ $\quad$ D: ℝ, R: ℝ

$y + 2 = t$

$\boxed{\begin{array}{l} x \geq 0 \\ y \in \mathbb{R} \end{array}}$ $\quad$ $\boxed{d}$

$\boxed{x = (y+2)^2}$

Parabola – Opens → $x \geq 0$

3) $x = \sqrt{t}$ $\quad$ D: $t \geq 0$, R: $x \geq 0$

$y = t$ $\quad$ D: $t \geq 0$, R: $y \geq 0$

$\boxed{b}$

$x = \sqrt{y}$

$\boxed{x^2 = y \; ; \; x \geq 0, y \geq 0}$

4) $x = \dfrac{1}{t}$ $\quad$ D: $t \neq 0$, R: $x \neq 0$

$y = t + 2$ $\quad$ D: $t \neq 0$, R: $y \neq 2$

$t = \frac{1}{x}$

$\boxed{y = \frac{1}{x} + 2 \quad \begin{array}{l} x \neq 0 \\ y \neq 2 \end{array}}$ Asymptotes

$\boxed{a}$

5) $x = \ln t$ $\quad$ D: $t > 0$, R: ℝ

$y = \dfrac{1}{2}t - 2$ $\quad$ D: $t > 0$, R: $y > -2$

$\boxed{f}$

$t = e^x$

$\boxed{y = \frac{1}{2}e^x - 2 \quad x \in \mathbb{R} ; y > -2}$

6) $x = -2\sqrt{t}$ $\quad$ $t \geq 0$, $x \geq 0$

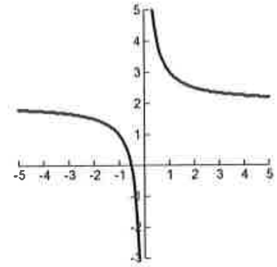
$y = e^t$ $\quad$ $t \geq 0$, $y \geq 1$

$x^2 = 4t$

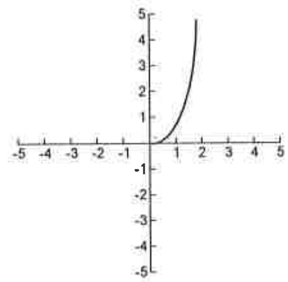
$\frac{x^2}{4} = t$

$\boxed{y = e^{\frac{x^2}{4}} \quad \begin{array}{l} x \geq 0 \\ y \geq 1 \end{array}}$ $\quad$ $\boxed{e}$

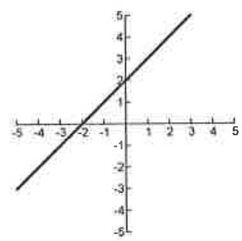
a)



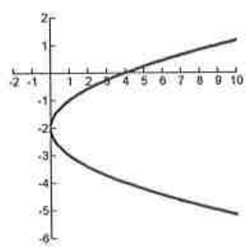
b)



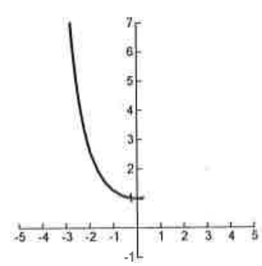
c)



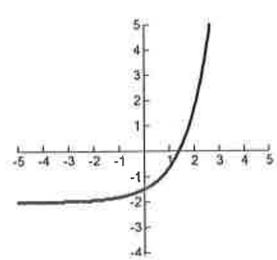
d)



e)



f)



- Sketch the curve represented by the parametric equations. Indicate orientation.
- Eliminate the parameter and write the corresponding rectangular equation. Indicate domain.

7) $x = t$ D: R R: R

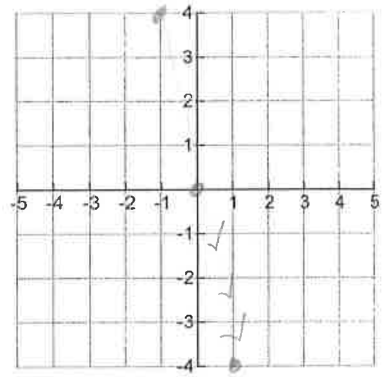
$y = -4t$ D: R R: R

$y = -4x$

D: R

R: R

t	0	1
x	0	1
y	0	-4



8) $x = 3t - 3$ D: R R: R

$y = 2t + 1$ D: R R: R

$\frac{x+3}{3} = t$

$y = 2\left(\frac{x+3}{3}\right) + 1$

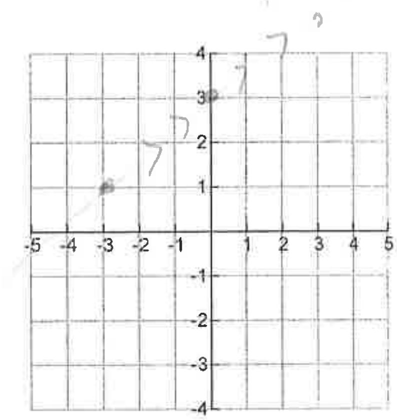
$y = \frac{2}{3}x + 2 + 1$

$y = \frac{2}{3}x + 3$

D: R

R: R

t	0	1
x	-3	0
y	1	3



9) $x = \frac{1}{4}t$ D: R R: R

$y = t^2$ D: R $y \geq 0$

$t = 4x$

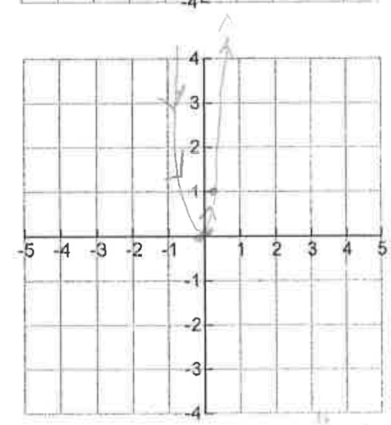
$y = (4x)^2$

$y = 16x^2$

$x \in R$

$y \geq 0$

t	0	1
x	0	$\frac{1}{4}$
y	0	1



10) $x = t + 2$ D: R R: R

$y = t^2$ D: R $y \geq 0$

$x - 2 = t$

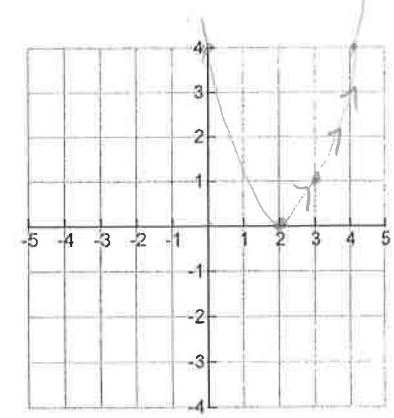
$y = (x-2)^2$

$x \in R$

$y \geq 0$

V (2, 0)

t	0	1
x	2	3
y	0	1



11) $x = 2t$ D: R R: R

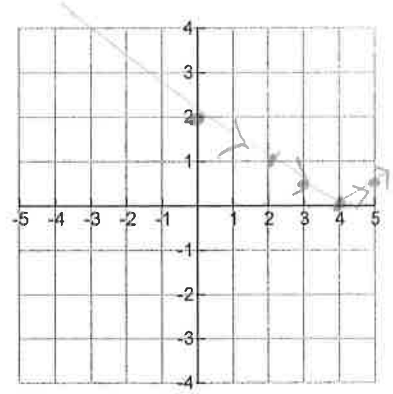
$y = |t-2|$ D: R R: $y \geq 0$

$t = \frac{x}{2}$

$y = \left|\frac{x}{2} - 2\right|$

$\boxed{y = \frac{1}{2}|x-4| \quad x \in \mathbb{R} \quad y \geq 0}$

t	0	1
x	0	2
y	2	1



12) $x = 3\cos\theta$ $0 \leq \theta < 2\pi$ $-3 \leq x \leq 3$

$y = 3\sin\theta$ $0 \leq \theta < 2\pi$ $-3 \leq y \leq 3$

$\cos\theta = \frac{x}{3}$ $\sin\theta = \frac{y}{3}$

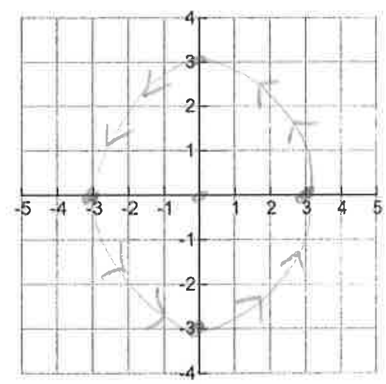
$\cos^2\theta + \sin^2\theta = 1$

$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{9} = 1$

$x^2 + y^2 = 9$ — O; r = 3

θ	0	$\frac{\pi}{2}$
x	3	0
y	0	3



13) $x = e^{-t}$ $x > 0$

$y = e^{3t}$ $y > 0$

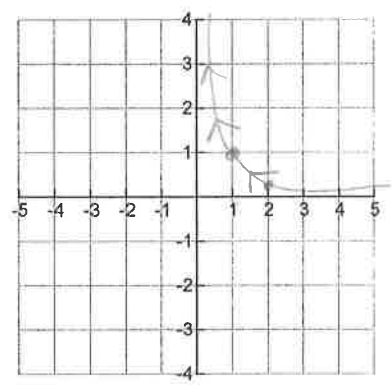
$\frac{1}{x} = e^{t}$

$y = (e^{t})^3$

$y = \left(\frac{1}{x}\right)^3$

$y = \frac{1}{x^3}$

t	0	1
x	1	e^{-1}, .367
y	1	e^3 20



14) $x = t^3$ $t > 0$ $x > 0$

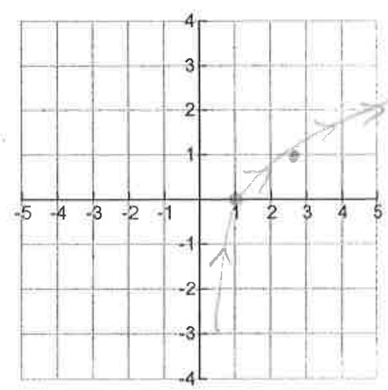
$y = 3\ln t$ $t > 0$ $y \in \mathbb{R}$

$x^{\frac{1}{3}} = t$

$y = 3\ln(x^{\frac{1}{3}})$

$y = \ln x$

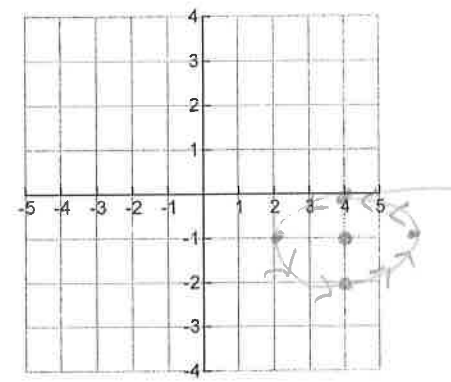
t	1	2
x	1	8
y	0	2.79



15) $x = 4 + 2\cos\theta$ $\quad 2 \le x \le 6$

$y = -1 + \sin\theta$ $\quad -2 \le y \le 0$

θ	0	$\frac{\pi}{2}$
x	6	4
y	-1	0



$\frac{x-4}{2} = \cos\theta$ $\qquad y+1 = \sin\theta$

$\left(\frac{x-4}{2}\right)^2 = \cos^2\theta$ $\qquad (y+1)^2 = \sin^2\theta$

$\left(\frac{x-4}{2}\right)^2 + (y+1)^2 = 1$

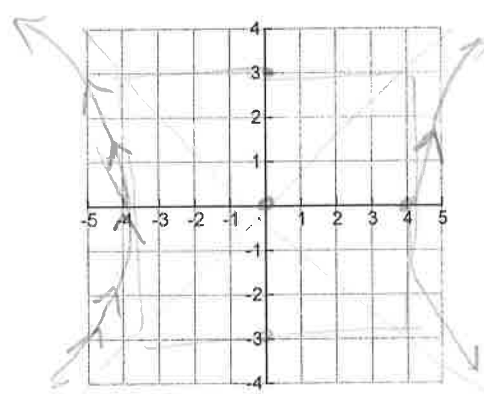
$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$

ELLIPSE
Horizontal MA
C (4, -1)
a = 2
b = 1

16) $x = 4\sec\theta$ $\quad x \ge 4 \quad x < -4$

$y = 3\tan\theta$ $\quad y \in \mathbb{R}$

t	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π
x	4	5.6	-5.6	-4
y	0	3	-3	0



$\frac{x}{4} = \sec\theta$ $\qquad \frac{y}{3} = \tan\theta$

$\frac{x^2}{16} = \sec^2\theta$ $\qquad \frac{y^2}{9} = \tan^2\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \frac{y^2}{9} = \frac{x^2}{16}$

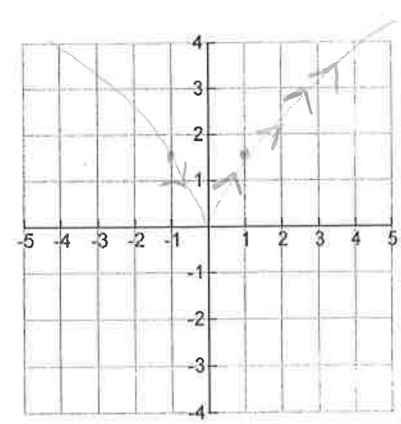
$1 = \frac{x^2}{16} - \frac{y^2}{9}$

Hyperbola
C (0,0) Horizontal TA
a = 4
b = 3

17) $x = \frac{t}{2}$

$y = \ln(t^2 + 1)$ $\quad y > 0$

t	0	2	-2
x	0	1	-1
y	0	1.6	1.6



$t = 2x$ $\qquad y = \ln((2x)^2 + 1)$

$y = \ln(4x^2 + 1)$