2.3 Real Zeros of Polynomial Functions

🙂 What will you learn? 🙂

- To use long division for polynomials
- To use synthetic division
- To use the Remainder Theorem
- To use the Rational Zero Test
- To use Descartes's Rule of Signs
- To use Upper and Lower Bound Rules

Long Division of Polynomials

You've done this before!

Given $f(x) = 6x3 - 19x2 + 16x - 4$

Use window : $-0.5 \le x \le 2.5$
$-0.5 \le y \le 0.5$

It appears that $x = 2$ is a zero of the function.
Therefore $(x - 2)$ is a factor.
Therefore there exists a 2nd degree polynomial factor $q(x)$ s.t.
$f(x) = (x - 2) \cdot q(x)$
Use Long Division to find $q(x)$

Long Division

$$x - 2 \;\big)\overline{\,6x3 - 19x2 + 16x - 4\,}$$

See p.123; exercise 1

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials s.t. $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$,
there exist unique polynomials $q(x)$ and $r(x)$ s.t.

$$\underset{\substack{\uparrow\\ \text{Dividend}}}{f(x)} = \underset{\substack{\uparrow\\ \text{Divisor}}}{d(x)}\,\underset{\substack{\uparrow\\ \text{Quotient}}}{q(x)} + \underset{\substack{\uparrow\\ \text{Remainder}}}{r(x)}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.
If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

Example

Remember ..... Write dividend & divisor in descending order
All powers must be accounted for (use "0" as a placeholder for missing powers)

$$x - 2 \,\overline{\big)\, 6x3 - 19x2 + 16x - 4}$$

Example 2 - Long Division of Polynomials

Divide $x3 - 1$ **by** $x - 1$

Remember ..... Write dividend & divisor in descending order
All powers must be accounted for (use "0" as a placeholder for missing powers)

Check

See p. 123; exercise 7

Example 3 - Long Division of Polynomials

Divide $2x4+ 4x3 -5x2+ 3x - 2$ **by** $x2+ 2x - 3$

See p. 123; exercise 9

Synthetic Division

You've done this before, too!

Divisor has to be in the form $x - k$
All powers have to acconted for!

Example 4 - Using Synthetic Division

Use synthetic division to divide $x4 - 10x2 - 2x + 4$ **by** $x + 3$

Exploration
Evaluate the the above dividend at $x = -3$

What do you notice???

See p. 123; exercise 19

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$,
the remainder is $r = f(k)$

Example 5 - Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$

$$f(x) = 3x3 + 8x2 + 5x - 7$$

See p. 124; exercise 31

The Factor Theorem

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$

Example 6 - Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x4 + 7x3 - 4x2 - 27x - 18$$

Algebraic **Graphical**

See p. 124; exercise 39

Using the Remainder in Synthetic Division

1. **The remainder gives the value of**________________________________
2. **If** $r = 0$________________________________
3. **If** $r = 0$ ______________________________

The Rational Zero Test

relates the possible rational zeros to the lead coefficieint and constant term

If the polynomial

$$f(x) = anx\ n + an\text{-}1xn\text{-}1 + \ldots + a2x2 + a1x + a0$$

has integer coefficients, every *rational zero* of *f* has the form

$$Rational\ Zero = \frac{p}{q}$$

where *p* and *q* have no common factor other than *1*, *p* is a factor of the constant term *a0*, and *q* is a factor of the leading coefficient *an* .

1. **List all the factors of constant term (numerator)**
2. **List all the factors of leading coefficient (denominator)**
3. **List all rational numbers s.t.** $\dfrac{\text{factors of constant term}}{\text{factors of leading coefficient}}$
4. **Use trial and error to determine which, if any, are actual zeros**

Example 7 - Rational Zero Test with Leading Coefficient = 1

Find the rational zeros of $f(x) = x3 + x + 1$

See p. 124; exercise 45

If Leading Coefficient $\neq 1$

Try

Programmable Calculator
Graphing Utility
Intermediate Value Theorem
Factor Theorem & Synthetic Division

Example 8 - Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x3 + 3x2 - 8x + 3$

See p. 124; exercise 47

Activities

1. Use Synthetic Division to determine if $(x+3)$ is a factor of

$$f(x) = 3x3 + 4x2 - 18x - 3$$

2. Divide using Long Division

$$\frac{4x5 - x3 + 2x2 - x2}{x + 1}$$

3. Use the Remainder Theorem to evaluate

$f(-3)$ for $f(x) = 2x3 - 4x2 + 1$

4. Use the Rational Zero Test to find all the possible rational zeros of

$$f(x) = 6x3 - x2 + 9x + 4$$

Example 9 - Finding Real Zeros of a Polynomial Function

Find all the real zeros of

$$f (x) = 10 x3 - 15x2 - 16x + 12$$

See p. 124; exercise 51

Other Tests for Zeros of Polynomials

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$

be a polynomial with real coefficients and $a_0 \neq 0$.

1. The umber of POSITIVE REAL ZEROS of *f* is *either* equal to the number of sign changes of $f(x)$ *or* less then that by an even integer.

2. The number of NEGATIVE REAL ZEROS of *f* is *either* equal to the number of sign changes if $f(-x)$ *or* less than that by an even integer.

Example : $x^3 - 3x + 2$

Example 10 - Using Descartes Rule of Signs

Describe the possible real zeros of

$$f(x) = 3x3 - 5x2 + 6x - 4$$

# of sign changes of $f(x)$:______

$$f(-x) =$$

# of sign changes of $f(-x)$:______

See p. 124; exercise 57

Upper & Lower Bound Rules

Let *f (x)* be a polynomial with *real coefficients* and a *positive leading coefficient.*
Suppose *f (x)* is divided by *x - c*, using synthetic division.

1. If *c > 0* and each number in the last row is either positive or zero,
   c is an upper bound for the real zeros of *f*

2. If *c < 0* and the numbers in the last row are alternately positive and negative
   (zero entries count as positive of negative), *c* is lower bound for the real zeros of *f*

Example 11 - Finding the Zeros of a Polynomial Function
Find the real zeros of

f (x) = 6x3- 4x2+ 3x - 2

See p. 125; exercise 67

2 more Hints:

1. If the terms of $f(x)$ have a common monomial factor, it should be factored before applying the tests in this section.

$$f(x) = x4 - 5x3 + 3x2 + x$$

2. If you are able to find all but 2 zeros of $f(x)$, you can always use the QUADRATIC FORMULA on the remaining quadratic factor.

$$f(x) = x4 - 5x3 + 3x2 + x = x(x - 1)(x2 - 4x - 1)$$