2.5 The Fundamental Theorem of Algebra

What will you learn?

- To use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function
- To find all the zeros of polynomial functions
- To find conjugate pairs of complex zeros
- To find zeros of polynomials by factoring

The Fundamental Theorem of Algebra

Carl Friedrich Guass (1777-1855)

If $f(x)$ is a polynomial of degree n, where $n>0$,
then f has a *least one zero* in the complex number system

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$,
f has precisely n linear factors

$$f(x) = an(x - c1)(x - c2)\dots(x - cn)$$

where c1, c2, . . . cnare complex numbers

Example 1 - Real Zeros of a Polynomial Function

Counting multiplicity, justify that the second degree polynomial function

$f(x) = x2 - 6x + 9 = (x - 3)(x - 3)$

has *exactly two* zeros : $x = 3$ **and** $x = 3$

See p. 140; exercise1

Example 2 - Real & Complex Zeros of a Polynomial Function

Justify that the third-degree polynomial function

$$f(x) = x3 + 4x = x(x2 + 4)$$

has *exactly three* zeros : $x = 0$, $x = 2i$ and $x = -2i$

See p. 140; exercise 3

Example 3 - Finding the Zeros of a Polynomial Function

Write

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

as the product of linear factors, and list all the zeros of *f*.

See p. 140; exercise 25

Conjugate Pairs

$a + bi$	$a - bi$

Complex Zeros occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*.
If $a + bi,\ b \neq 0$, is a zero of the function,
the conjugate $a - bi$ is also a zero of the function.

Example 4 - Finding a Polynomial with Given Zeros

Find a *4th-degree* polynomial function with real coefficiens that has *-1, -1 and 3i* as zeros

See p. 141; exercise 37

Factoring a Polynomial

Factors of a Polynomial

Every polynomiial of degree *n> 0* with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros

A quadratic factor with no real zeros is said to be prime or irreducible over the reals

(Not the same as being irreducible of the rationals)

Example 5 - Factoring a Polynomial

Given $f(x) = x^4 - x^2 - 20$

Write the polynomial as :

a.) the product of factors that are irreducible over the *rationals*

b.) the product of linear factors and quadratic factors the are irreducible over the *reals*

c.) completely factored form

See p. 141; exercise 41

Activity

1. **Write as a product of linear factors:** $f(x) = x^4 - 16$

2. **Find the 3rd-degree polynomial with integer coefficients that has** $2, 3 + i$ **and** $3 - i$ **as zeros**

3. **Write the polynomial** $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$ **in completely factored form**

   (Hint: one factor is $x^2 - 2x - 2$ **)**

Example 6 - Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x4 - 3x3 + 6x2 + 2x - 60$$

given that $1 + 3i$ **is a zero of** f

Algebraic

Graphical

See p. 141; exercise 47