Sec 8.4:

Improper Integrals

Evaluate by using your calculator.

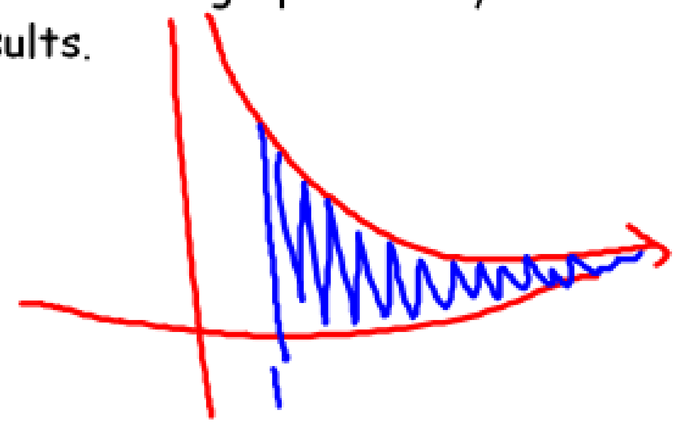
$\int_1^{100} \frac{1}{x}\,dx = $ 4.60517 $\quad \int_1^{1000} \frac{1}{x}\,dx = $ 6.90776 $\quad \int_1^{1,000,000} \frac{1}{x}\,dx = $ 13.8155

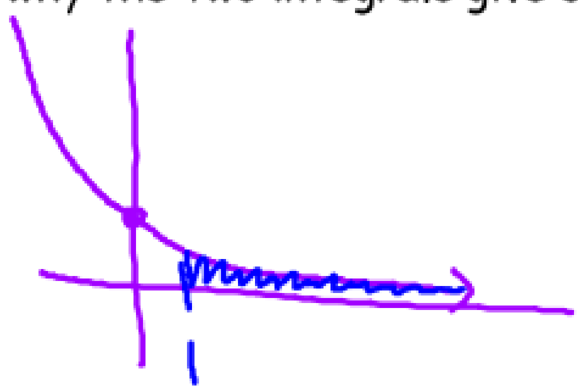
Do you think $\int_1^{\infty} \frac{1}{x}\,dx$ converges or diverges? ∞

$\int_1^{100} e^{-x}\,dx = $ .367879 $\quad \int_1^{1000} e^{-x}\,dx = $ .367879

Do you think $\int_1^{\infty} e^{-x}\,dx$ converges or diverges? $= .367\ldots = \frac{1}{e}$

Look at each graph and try to determine why the two integrals give such different results.





Integrals such as the ones below are called improper integrals. They are evaluated by rewriting the integral as a proper integral and then using limits.

Improper Integrals with an infinite bound:

$$\int_a^{\infty} f(x)\,dx \qquad \int_{-\infty}^{a} f(x)\,dx \qquad \int_{-\infty}^{\infty} f(x)\,dx$$

Evaluate $\int_1^{\infty} \frac{1}{x}\,dx$ dne (diverges)

Evaluate $\int_1^{\infty} e^{-x}\,dx = \frac{1}{e}$

Evaluate $\int_{-\infty}^{0} e^{\frac{x}{4}}\, dx$

$$\lim_{a \to -\infty} \int_a^0 e^{x/4}\, dx = \lim_{a \to -\infty} 4e^{x/4} \Big|_a^0$$

$$\lim_{a \to -\infty} \left[4 - 4\underbrace{e^{a/4}}_{\frac{1}{e^{+\infty/4}}} \right] = \boxed{4}$$

Evaluate $\int_1^{\infty} (1-x)e^{-x}\,dx$

$$\lim_{a\to\infty} \int_1^a (1-x)e^{-x}\,dx$$

$u = 1-x \qquad dv = e^{-x}$

$du = -1 \qquad v = -e^{-x}$

$$\lim_{a\to\infty}\left[-e^{-x}(1-x) - \int_1^a e^{-x}\,dx\right] = \lim_{a\to\infty}\left[\cancel{-e^{-x}} + xe^{-x} + \cancel{e^{-x}}\,\Big|_1^a\right]$$

$$\lim_{a\to\infty}\left[xe^{-x}\Big|_1^a\right] = \lim_{a\to\infty}\left[ae^{-a} - 1e^{-1}\right]$$

$\frac{\infty}{\infty}$

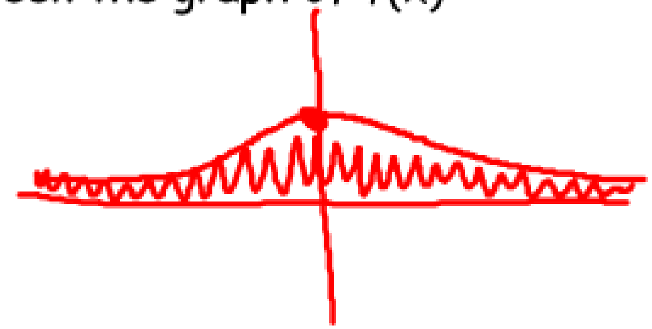
$$\lim_{a\to\infty}\left[\frac{a}{e^a} - \frac{1}{e}\right] = \boxed{-\frac{1}{e}}$$

$$\frac{1}{e^a}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Let R be the region in the first and second quadrants between the graph of f(x) below and the x-axis. Find the area of the region.

$$f(x) = \frac{1}{1+x^2}$$



$$A = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx = 2\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$2\lim_{a\to\infty} \int_{0}^{a} \frac{1}{1+x^2}\,dx = 2\lim_{a\to\infty}\left[\tan^{-1}(x)\Big|_{0}^{a}\right]$$

$$2\lim_{a\to\infty}\left[\tan^{-1}(a) - 0\right] = 2\left[\frac{\pi}{2} - 0\right] = \boxed{\pi}$$

Evaluate by using your calculator.

$\int_{.01}^{1} x^{-1/3}dx =$ 1.43038 $\quad \int_{.001}^{1} x^{-1/3}dx =$ 1.485 $\quad \int_{.0001}^{1} x^{-1/3}dx =$ 1.49677

Do you think $\boxed{\int_{0}^{1} x^{-1/3}dx}$ converges or diverges? = 1.5

$\int_{.01}^{1} x^{-3}dx =$ 4999.5 $\quad \int_{.001}^{1} x^{-3}dx =$ 500,000 $\quad \int_{.0001}^{1} x^{-3}dx =$ 5,000,000

Do you think $\boxed{\int_{0}^{1} x^{-3}dx}$ converges or diverges?

Look at each graph and try to determine why the two integrals give such different results.

The second type of improper integral is one where there is a discontinuity at one of the limits or somewhere between the limits. For these you must again rewrite them using limits.

Evaluate $\int_0^1 x^{-1/3}\,dx$

Evaluate $\int_0^1 x^{-3}\,dx$

Evaluate

$$\int_0^{27} \frac{1}{\sqrt[3]{27-x}}\,dx$$

disc @ $x=27$

$$\lim_{a\to 27^-} \int_0^a (27-x)^{-1/3}\,dx = \lim_{a\to 27^-} -\frac{3}{2}(27-x)^{2/3}\Big|_0^a$$

$$= \lim_{a\to 27^-} \left[-\frac{3}{2}(27-a)^{2/3} - -\frac{3}{2}(27)^{2/3}\right]$$

$$\frac{3}{2}(9) = \boxed{\frac{27}{2}}$$

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}}dx = \int_0^1 \frac{1}{(x-1)^{2/3}}dx + \int_1^3 \frac{1}{(x-1)^{2/3}}dx$

$$\lim_{a\to 1^-} \int_0^a (x-1)^{-2/3}dx + \lim_{b\to 1^+} \int_b^3 (x-1)^{-2/3}dx$$

$$\lim_{a\to 1^-}\left[3(x-1)^{1/3}\Big|_0^a\right] + \lim_{b\to 1^+}\left[3(x-1)^{1/3}\Big|_b^3\right]$$

$$\lim_{a\to 1^-}\left[3(a-1)^{1/3} - -3\right] + \lim_{b\to 1^+}\left[3\sqrt[3]{2} - 3(b-1)^{1/3}\right]$$

$$\boxed{3 + 3\sqrt[3]{2}}$$

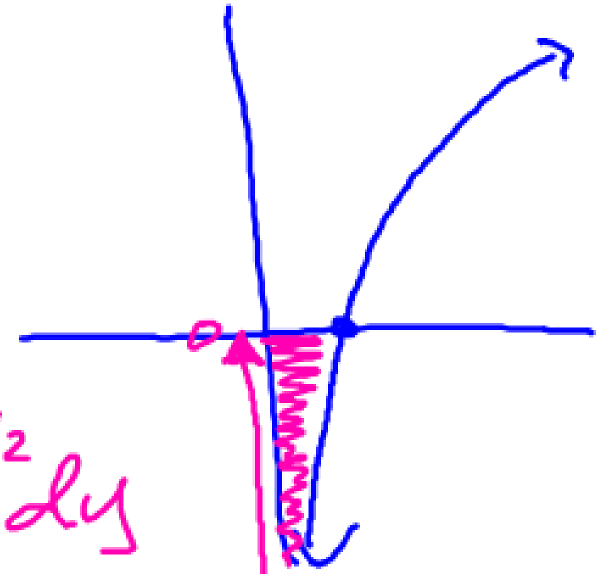
Let R be the unbounded region in the fourth quadrant between the graph of f(x) given below and the y-axis. Find the area of R.

$$f(x) = 2\ln x$$

$y = 2\ln x$

$\frac{y}{2} = \ln x$

$e^{y/2} = x$

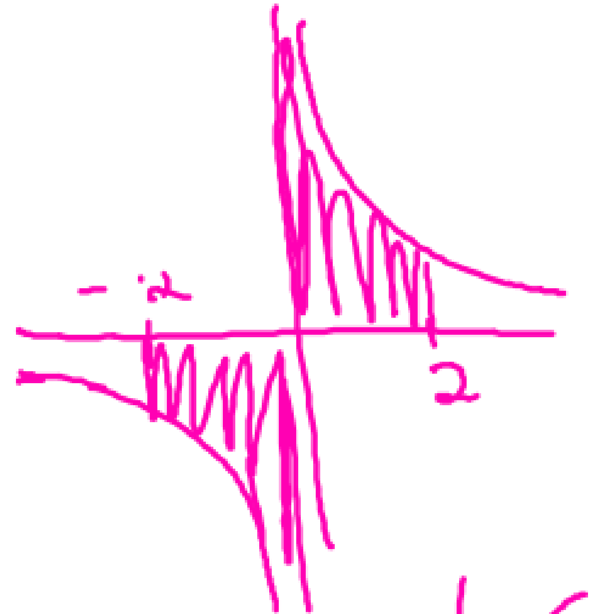


$$A = \int_{-\infty}^{0} e^{y/2}\,dy = \lim_{a\to-\infty} \int_{a}^{0} e^{y/2}\,dy$$

$$\lim_{a\to-\infty}\left\{2e^{y/2}\Big|_{a}^{0}\right\} = \lim_{a\to-\infty}\left[2 - 2e^{a/2}\right]^{-\infty} = \boxed{2}$$

Let R be the unbounded region in the first and third quadrants between the graph of f(x) below and the x-axis over the interval [-2, 2]. Find the area of R.

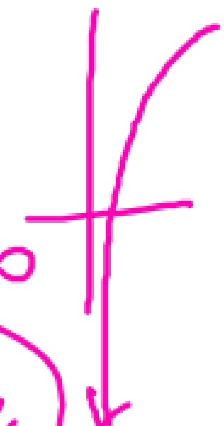
$$f(x)=\frac{1}{x}$$



$$A=\int_{-2}^{2}\frac{1}{x}\,dx=\int_{-2}^{0}\frac{1}{x}\,dx+\int_{0}^{2}\frac{1}{x}\,dx$$

$$=2\int_{0}^{2}\frac{1}{x}\,dx=2\lim_{a\to 0}\int_{a}^{2}\frac{1}{x}\,dx$$

$$2\lim_{a\to 0}\left[\ln x\Big|_{a}^{2}\right]=2\lim_{a\to 0}\left[\ln 2-\overset{-\infty}{\ln a}\right]=\infty$$



diverge