Lagrange Error Bound

or

Taylor's Theorem Remainder

When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

In other words: Remainder = Function - Polynomial

$$R_n(x) = f(x) - P_n(x)$$

Taylor's Theorem: If a function f is differentiable through order $n+1$ in an interval containing c, then for each x in the interval, there exists a number z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{(n)!}(x-c)^n + R_n(x)$$

where the remainder $R_n(x)$ (or error) is given by $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

Basically, we are looking at the first term left off of the polynomial and using the max value of the derivative within the given interval. The max value of the derivative will either be given to you, or you will need to find it by using curve sketching techniques!

Let f be a function with 5 derivatives on the interval [2, 3]. Assume that $\left|f^{(5)}(x)\right| < 0.2$ for all x in the interval [2, 3] and that a fourth-degree Taylor polynomial for f at $c=2$ is used to estimate $f(3)$.

How accurate is this approximation?

Suppose $P_4(3)=1.763$. Find an interval in which $f(3)$ must lie. Could $f(3)=1.768$? Could $f(3)=1.764$? Why or why not?

Find the fifth-degree Maclaurin polynomial, $P_5(x)$ for $f(x)=\sin x$. Then use your polynomial to approximate $\sin(1.45)$ and use Taylor's Theorem to find the maximum error for your approximation.

Find an interval [a, b] such that $a \le \sin(1.45) \le b$. Could $\sin(1.45)=0.977$? Why or why not?

What is the actual value of $\sin(1.45)$? Find $|f(1.45)-P_5(1.45)|$ and tell what it represents.

Write a second-degree Taylor polynomial, $T_2(x)$, for the function $f(x) = x^{5/2}$, centered at $a = 4$. Approximate the value of $f(4.3)$ by finding $T_2(4.3)$

Find the maximum possible size of the error for the approximation found by using the Lagrange Error Bound.

Find the actual value of f(4.3) and then find $|f(4.3) - T_2(4.3)|$. Is this within your maximum error?

The function f has derivatives of all orders for all real numbers, x. Assume that

$$f(2)=6 \quad f'(2)=4 \quad f''(2)=-7 \quad f'''(2)=8$$

Write the third-degree Taylor polynomial for f about $x=2$ and use it to approximate $f(2.3)$.

The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 9$ for all x in the closed interval [2, 2.3]. Use the Lagrange error bound on the approximation of $f(2.3)$ to find an interval [a, b] such that $a \leq f(2.3) \leq b$. Could $f(2.3)=6.922$?