Section 9.4:
Other Tests for Convergence

nth Term Test

Direct Comparison Test

Ration Test

Geometric Series Test

We are going to look a little closer at the radius of convergence. So far, we have looked at what x-values will give a good approximation of our function, but we have not looked very closely at the endpoints. To do that, we need to know a few more tests for proving convergence.

When do you need to test for convergence of an infinite series?

- What is the radius or interval of convergence for a particular series?
- Is it possible to evaluate f(a) using a Taylor Series expansion with some amount of accuracy?
- Which of the following series converge/diverge?
- For what values of k does a series whose nth term is a function of k converge?

We will start by reviewing the Geometric Series Test and the Ratio Test. Then we will learn two more today: the nth Term Test for Divergence and the Direct Comparison Test.

Geometric Series Test

A geometric series is in the form $\sum_{n=0}^{\infty} a_1(r)^n$ or $\sum_{n=1}^{\infty} a_1(r)^{n-1}$ $\quad a \neq 0$

- The geometric series *diverges* if $|r| \geq 1$
- The geometric series *converges* to the sum $S = \dfrac{a_1}{1-r}$ if $|r| < 1$

Determine if the following series converge or diverge. If they converge, what is the sum?

$$\sum_{n=0}^{\infty} 5(1.067)^n$$

$$\sum_{n=1}^{\infty} -\left(\frac{1}{8}\right)^n$$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$

3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

*Series with factorials and exponential functions work especiall well with the Ratio Test.

Determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$$

nth Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: This test can only be used for *divergence*. If $\lim_{n\to\infty} a_n = 0$ then this test tells us nothing and we need to use another test to check for convergence/divergence.

Determine if the following series diverge.

$$\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$$

$$\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$$

$$\sum_{n=1}^{\infty} \frac{3^n-2}{3^n}$$

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+3}$$

Direct Comparison Test:

Given $a_n \geq 0$ and $b_n \geq 0$

- If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges

  (Less than a convergent converges)

- If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges

  (More than a divergent diverges)

Determine if the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n - 5}$$

What about these? Find the radius and interval of convergence of the following geometric series.

$$\sum_{n=0}^{\infty}\left(\frac{x+1}{2}\right)^n$$

$$\sum_{n=0}^{\infty}(x+e)^n$$