CANTOR'S TRANSFINITE NUMBERS

Introduction to Philosophy Orange Coast College
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The transfinite numbers are cardinal numbers (used for counting) that specify the relative sizes of infinite collections (or sets) of things. Cantor decided to use the letters from the Hebrew alphabet for the basis of the symbols used to label the transfinite numbers. Hence, he choose the first letter in the Hebrew alphabet, $\aleph$, called an aleph. The number $\aleph_0$, called aleph null (or aleph zero) specifies the size of any infinite set that can be put into one to one (1-1) correspondence with the natural numbers. A 1-1 correspondence is when every member of one set can be paired up with every member of the other set so that for each member in one set there is a corresponding unique member in the other set and vice versa. For example, the natural numbers can be put into 1-1 correspondence with a subset of itself, namely the set of even numbers. For every natural number in the set each n is put into 1-1 correspondence with element 2n in the set of even numbers.

$$
\begin{array}{ll}
\text{Natural numbers} = & \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\} \\
& \;\;| \;\;\; | \;\;\; | \;\;\; | \;\;\; | \;\;\;\; | \;\;\;\; | \;\;\;\; | \\
\text{Even numbers} = & (2, 4, 6, 8, 10, 12, 14, 16, \ldots\}
\end{array}
$$

Hence, the natural numbers themselves can be said to contain aleph null, $\aleph_0$, **number of members or elements since it can be put into a 1-1 correspondence with itself. For every number n in the first set can be placed into a 1-1 pairing with number n in the other set.**

$$
\begin{array}{l}
\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\} \\
\;| \;\;\; | \;\;\; | \;\;\; | \;\;\; | \;\;\; | \;\;\; | \;\;\; | \\
\{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}
\end{array}
$$

Similarly, since the set of all of the even numbers, {2, 4, 6, 8, . . . }, and the set of all of the odd numbers, {1, 3, 5, 7, . . .}, etc. can each be put into one to one correspondence (1-1) with the set of natural numbers, they too have an $\aleph_0$ number of elements.

Cantor wondered if all sets have the same cardinality of $\aleph_0$. Intuitively it would seem that they do since any infinite set can be put into a 1-1 correspondence with a proper subset of itself.

To explain this more clearly, a few definitions are in order.

A *set* is any collection of objects with a well defined condition for set membership. For example, there is the set of chairs that contains all and only objects that are chairs, {desk chair, straighbacked chair, armchair, etc. }.

A *subset* is a set formed using only the objects from one particular set and the subset cannot contain any members that are not found in the set from which the subset is formed. For example, a subset may be formed from a set {A, 3, C} (call it the original set) by choosing the element A and forming the subset {A}. A set that contained a B such as {A, B, 3} would not be a subset of the original set {A, 3, C} since B is not a member of this set. Note also that every set is a subset of itself, hence the set {A, 3, C} is a subset of the original set.

A *proper subset* is a subset that does not contain every member from the set from which it is formed. So in the above examples, the unit set {A} is a proper subset of the original set, however the set {A, 3, C} is not a proper subset of the original set.

The concept of proper subset logically entails that for all finite sets (sets containing only a finite and not an infinite number or objects or elements) any proper subset will also have fewer members and therefore be smaller in numerical size since it necessarily will contain fewer members than the set from which it has been formed. Removing any element from a finite amount will always result in a lesser amount.

Surprisingly, this is not true for infinite sets that a proper subset must always be smaller in numerical size. Galileo pointed out that one can even remove an infinite number of elements from a infinite set and still put the two sets into 1-1 correspondence thereby proving the two sets are numerically equal in size. For example, if we remove all of the odd numbers from the set of natural numbers we are still left with an infinite number of even numbers. Because the set of even number are all contained in the set of natural numbers and does not contain any elements not found in the set of natural numbers, the set of even numbers is a subset of the natural numbers. Furthermore, because the set of even numbers does not contain all of the natural numbers it is a proper subset of the natural numbers. Still, both the set of natural numbers and even numbers can be put into 1-1 correspondence:

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{1, 2, 3, 4,  5,  6,  7,  8, . . .}
 |  |  |  |   |   |   |   |
(2, 4, 6, 8, 10, 12, 14, 16, . . .}
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Galileo found this result so unusual and odd that it has come to be known as *Galileo's paradox*. How can what is only part of a whole be the same in numerical size as all of the whole. This seems to violate common sense.

Georg Cantor's reaction to Galileo's paradox was not to see it as a paradox, but only as a revelation that infinite sets can have different types of properties from that of those had by finite sets. The parts will always be smaller than the wholes for any finite sets, but this does not hold true for infinite sets.

Cantor's attitude was that one can use Galileo's paradox where it is possible to put a proper subset of the set of natural numbers, all of the even numbers, into 1-1 correspondence with all of the members of the original set (including both odd and even numbers) and that this can be used as a criteria for when a set has or contains an infinite number of members. Therefore, so much for common sense which has been shown to be mistaken about the part/whole relationships of infinite sets.

Cantor was able to prove that a *power set* of any set will always be a numerically larger set than the set from which the power set has been formed.

As was explained earlier, any set that can be put into a 1-1 correspondence with the natural numbers has an aleph null number of elements. When we put these two ideas together it follows that the power set of the infinite set of natural numbers will necessarily have to be a larger set and have more than the infinite number of members or elements contained in the natural numbers. Since the natural numbers contain an aleph null number of elements, and the power set of the natural numbers must contain a larger number of elements than aleph one, the size of the power set of the natural numbers is aleph one. The power set of this new aleph one size set must also have a larger number of elements and this set contains an aleph two number of elements, and so on through an infinity of larger power sets each with a correspondingly larger aleph number.