Rational Numbers & Real Numbers

A rational number, in mathematics, is one which can be represented by a fraction, the ratio of two integers (the one which does the dividing, the denominator, cannot be 0). The rational numbers were introduced to solve the problem that there were equations of the form a/b = ? with no solutions. It is usual to write rational numbers 'in their lowest terms', meaning that the two numbers in the fraction cannot have any integers which divide both of them. One of the problems which most distressed the Greeks was that there are still equations with no rational number solutions (the example they discovered that there was no rational number whose square is 2, which means that there is no solution of the equation x2 - 2 = 0). This problem, which the Greeks did not attempt to solve, was overcome by the introduction of the algebraic numbers (this was not possible until systematic notation for polynomials was introduced).

It may seem to be difficult to think of numbers which are not rational, but in fact, very few numbers really are. One of the first controversial results of the set theory of Georg Cantor (1845 - 1918) was that the number of rational numbers was not the same as the number of real numbers. (This was controversial because mathematicians felt that any idea that there were different sizes of infinity was impossible to consider.) Cantor showed that the rational numbers were countable (that is, could be counted or written in a list) whereas the real numbers were not. The proof that the rational numbers were countable relies on a famous 'diagonalisational argument', in which he gave a way in which they could be listed. The rationals are written in an infinite square by writing all those with denominator 1 in the top row, as 0/1, 1/1, 2/1 and so on, then all those with denominator 2 on the next row, and so on. The way to list them is to start at the top, left-hand corner, to go one to the right (to 1/1), then diagonally down and to the left (to 0/2), then down (to 0/3) and then diagonally up and to the right (to 1/2) and again (to 3/1); then to 4/1 and diagonally down and right again and so on. Every number will at some point be included in such a list. SMcL

The real numbers represent almost the final step in the various expansions of the number system from the integers to the rational numbers to the algebraic numbers to the real numbers and the complex numbers. The real numbers form a field, they have an ordering, and they also have the property of 'completeness'. This means that if there is any set X of real numbers such that they have an upper bound (a number that is bigger than everything in X), then they have a least upper bound (an upper bound which is smaller than any other). For example, suppose X is the set of numbers whose square is greater than 2. This has an upper bound (3 is bigger than anything in the set) and so has a least upper bound, which will be the square root of 2. The real numbers are (apart from just renaming the numbers) the only mathematical structure which is a complete ordered field.

The real numbers represent a considerable increase in size from the algebraic numbers. The algebraic numbers are countable (can be counted, or written in a list), whereas the real numbers cannot. This fact was first discovered by Georg Cantor (1845 - 1918), and was controversial because mathematicians felt that saying a set was infinite was all that could be said, there were no degrees of infinite (see set theory).

Why can the real numbers not be listed? Suppose that there is a way that they can be listed. Then there is a way of listing the numbers between 0 and 1 (as infinite decimals). But then there is a number which is not on the list: the one which in the nth decimal place has 1 if the nth decimal place of the nth number on the list is 2, and 2 if it is not. It is different from every number on the list, in the nth decimal place, and so no such list can be

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5. Rational Numbers

Infinity

Before the 19th century, the idea of the infinite was dismissed by mathematicians. Although they knew that there were infinite sets, they felt that nothing interesting could be said about them. Something was either infinite or not, and that was all that could be said.

Towards the end of that century, the German mathematician Georg Cantor (1845 - 1918) began to think somewhat about the idea of the infinite in mathematics. He tried, for example, to come up with a definition of the concept of the infinite. (Previously, such definitions as 'bigger than any number' were used, but that is in fact a circular definition, because the set of numbers is infinite, so that it amounts to a definition of infinite as 'as big as infinite'.) He also set out to categorize infinite sets.

One of the first things that Cantor realized was that some infinite sets were bigger than others. He defined when two sets were to be of the same size: his definition was based on the intuitive behaviour of the numbers which were already familiar. Two sets are 'equinumerous' (equal in number) if there is a mapping between them which is a bijection, that is, a function where the two non-equal elements in the first set have non-equal images, and where every element in the second set has an element which the function maps to it. For example, the sets 1,2 and 5,7 are equinumerous, because the function mapping 1 to 5 and 2 to 7 is a bijection.

The smallest infinite set is that of all the natural numbers, and any set which is equinumerous with them is called countable or enumerable, because it is possible to write any such set as an infinite list (as the bijection between it and the natural numbers effectively gives you a first element, and a second element, and so on). Many kinds of numbers are countable, such as the integers and the rational numbers, while others are not (the way that the real numbers are shown to be uncountable is in algebraic numbers). It came as a big shock that there were such things as uncountable sets; previously it had seemed that you must be able to list the elements of any set. The reaction was so strong that many mathematicians condemned Cantor's results.

The same ideas give a non-circular definition of the infinite. An infinite set is defined to be one which is equinumerous with some subset of itself (other than the whole thing). For example, the set of natural numbers is equinumerous with the set of even numbers.

Today, however, the infinite is very much part of mathematics, and much of the work in set theory in this century has been to do with the various properties of infinite sets; see axiom of choice for a discussion of one of the most important. SMcL

Further reading R. Rucker, *Infinity and the Mind.*

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xreferences

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2. Rational Numbers Bloomsbury Guide to Human Thought
3. Set Theory Bloomsbury Guide to Human Thought
4. Intuitionism Bloomsbury Guide to Human Thought
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Published in the *St. John's Review*, XLIV, 2 (1998) 35-59.

This crash course is designed to stand alone. But it also functions as the appendix to my essay, Infinite Reflections.

A Crash Course in the Mathematics Of Infinite Sets

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Don't be surprised if this is easier than you thought. Set theory requires no algebra or calculus. It is much more primitive than those branches of mathematics, and rests on very simple notions. Moreover, the proofs will be unusually short and uncomplicated.

What will be difficult? Most of the results we will prove depend critically on those that came before; but I cite the needed prior theorems by number to make this kind of back-tracking easier. The notation may be new, and for many people unfamiliar notation raises the hair on the back of the neck. But most of the notation may be ignored. I include it mainly so that if you read further on this subject, you will be equipped. I honestly don't think the compressed exposition needed for a crash course increases the difficulty —in part because I've been more long-winded than most mathematicians, and in part because some compression and conciseness helps keep all the relevant ideas in the head at the same time, which aids comprehension. Some of the proofs, short and simple as they are, will make you dizzy. But that's part of the amazing phenomenon to be savored, not a difficulty to lament.

To begin:

Almost a definition. Intuitively, a *set* is a collection of elements.

- The intuitive notion of a set leads to paradoxes, and there is considerable mathematical and philosophical disagreement about how best to refine the intuitive notion. Fortunately, none of the disagreements or refinements matters for our purposes here. I only bring up this complexity so that you'll accept the intuitive notion in place of a refined definition for the purposes of this crash course.
- Notation. When we want to list the members of a set, we use curly brackets. So if set S contains elements A, B, and C, then we say S = {A, B, C}.
- The *null set* is the empty set or the set with no members. Notation: Ø. Hence, Ø = {}.

Abbreviation. For *if and only if* I will sometimes write simply *iff*.

Definition. Set A is a *subset* of set B iff all the members of A are also members of B.

- Notation. $A \subseteq B$.
- It follows from this definition that every set is a subset of itself.

Definition. Set A is a *proper subset* of set B iff all the members of A are also members of B, but not all the members of B are members of A.

- Notation. $A \subset B$.
- It follows from this definition that no set is a proper subset of itself.

Definition. The *cardinality* of a set is the number of members it contains.

- Notation. The cardinality of set S is |S|. For example, if S = {A, B, C}, then |S|=3.
- Hence while S is a set, |S| is a number. When S is an infinite set, |S| will be an infinite

- We will assume *the power set axiom*, i.e. that all sets have power sets.

Reminder. The *natural numbers* are the whole positive numbers (sometimes called the "counting numbers"), including zero: 0, 1, 2, 3 ....

- This is really a definition, but by calling it a "reminder" I'm hoping to get on your good side.
- Notation. The set of natural numbers is designated by **N**.
- Notation. The number of natural numbers is designated by $\aleph_0$. "$\aleph$" is the first letter of the Hebrew alphabet, pronounced *Aleph*. "$\aleph_0$" is pronounced *Aleph-null* or *Aleph-nought*. We will justify the zero subscript when we prove that no infinite set has a smaller cardinality than the set of natural numbers (Theorem 6).
- Hence $\aleph_0 = |\mathbf{N}|$, by definition.
- Now you know how many natural numbers there are: $\aleph_0$. But this is not profound. So far we've only invented a name (numeral) for the number of natural numbers.

Definition. A set is *countable* iff its cardinality is either finite or equal to $\aleph_0$. A set is *denumerable* iff its cardinality is exactly $\aleph_0$. A set is *uncountable* iff its cardinality is greater than $\aleph_0$.

- The null set is countable. The finite set, {A, B, C}, is countable. The infinite set, **N**, is countable and denumerable. Sets with a larger cardinality than **N** are uncountable.

Definition. A *transfinite number* or *transfinite cardinal* is the cardinality of some infinite set.

- If we use the term "infinite" in a restricted and precise way, then "transfinite" is just a synonym for it. We could avoid fancy new terms to prevent confusion. However, "infinite" has many imprecise and non-technical uses —for example, the *infinite* setting on a camera's range-finder— so it often helps to use a technical term to avoid ambiguity.

Reminder. The *integers* are the natural numbers plus their negative counterparts, ...-3, -2, -1, 0, 1, 2, 3....

- Notation. The set of integers is designated by **Z**.

Reminder. The *rational numbers* are the integers plus the rational fractions (those that can be expressed as the ratio of two integers).

- Notation. The set of rational numbers is designated by **Q**.
- For example, 0.75 is a rational fraction because we can express it as the ratio of two integers, namely, 3/4. Therefore it is a rational number.
- The *irrational numbers* are the fractions that are not rational numbers, both positive and negative. For example, we can prove that pi (3.14159...) cannot be expressed as the ratio of two integers. Therefore it is an irrational number.

Reminder. The *real numbers* are the rational numbers plus the irrational numbers.

- Notation. The set of real numbers is designated by **R**.

We started with the natural numbers, then added infinitely many negative whole numbers to get the integers, then added infinitely many rational fractions to get the rationals, and then added infinitely many irrational fractions to get the reals. It's tempting to conclude that with each infinite addition we increased cardinality, or in short:

Denumerable and non-denumerably infinite sets

Introduction to Philosophy
Dr. David C. Ring
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What are the differences between denumerable and non-denumerably infinite sets?

An *infinite set* is one that contains an infinite number of elements or members. This, by definition, is not a finite amount. A *finite set* is defined as one where it is possible to put the members or elements in the finite set into a one to one correspondence (1-1) from 1 (the first or lowest natural number) up to some specific natural number n with each and every member contained in the finite set.

An *infinite set* therefore is one where there is no natural number n that can be used to specify the number of elements contained in the infinite set. By adding the number one to any natural number one produces the next highest number.

So, the set of *positive whole integers* are the set of natural numbers starting with 1, then 2, then 3, 4, 5, 6, 7, . . . and so on.

Any set that can be put into one to one correspondence with all of the natural numbers is said to be a *denumerably infinite set*. This means, among other things, that the set is countable using only the natural numbers. Since every set must be in one to one correspondence with itself the set of natural numbers is, of course, a denumerably infinite set.

So too the set of all even numbers, {2, 4, 6, 8, 10, . . . } is denumerable since this set can be put into 1-1 correspondence with ALL of the natural numbers. The set of all odd numbers {1, 3, 5, 7, . . . } is also a denumerably infinite set for the same reasons.

When Georg (prounced "Gay-Org") Cantor was able to PROVE that the set of real numbers, the decimal numbers such as .333, cannot be put into 1-1 correspondence with the natural (1, 2, 3, . . .) or rational numbers (the fractions), this proved that one infinite set can have more members than another infinite set. Infinite sets come in different orders of infinity or in different numerical sizes, even though they are both infinite sets, they do not have the same number of members. But then some sets must be non-denumerably infinite since they cannot be put into 1-1 correspondence with all of the natural numbers.

How did Cantor prove that the real numbers could not be put into 1-1 correspondence with the natural numbers?

The answer is to suppose that you have succeeded in putting them into a 1-1 correspondence and then prove that this is not possible. Such a 1-1 correspondence could be said to possibly look like this:

1 = .33333333... (with 3's repeating to infinity)
2 = .66666666... (with 6's repeating to infinity)
3 = .1415962... (with randomly different numbers)
4 = etc.
and so forth.

Imagine now finding a decimal (real) number that cannot possibly be located anywhere on the list. You can *create* such a number by imagining the following procedure. We go to the first number in the list in the first decimal place (in this case it is a 3) and say .5 by adding 2 onto the number 3 in that same first decimal place. Next we go to the second number in the list in the second decimal place (in this case a 6) and continue on with our number that we are constructing that cannot possibly be located anywhere in the list and produce .58 by adding 2 onto the 6 in that second decimal place. We go to the third number and change the number 1 in the third decimal place by adding 2 onto it to produce our number that cannot be in the list .583 and we do this to infinity.

The infinitely long decimal number that starts .583 . . . is different from the first number in the first decimal place. It is different from the second number in the list in the second decimal place. It is different from the third decimal number in the third decimal place and so on. Yet it is still a number greater than zero and less than one that cannot be found on the list.

Since the number we constructed cannot possibly be found in the list this proves there are more real numbers than there are natural numbers. Every natural number has been paired up with a proper subset of the real numbers, but not every real number has been paired with a rational numbers since the number we have constucted using Cantor's diagonalization technique cannot be found on the list given how the number was constructed.

is that we can leave out some of the members of one list and *still* find a one-to-one correspondence between the two lists!) In a similar, but somewhat more complicated way, we can set up a one-to-one correspondence between the fractions and the integers. (For this we can adapt one of the ways of representing *pairs* of natural numbers, the numerators and denominators, as single natural numbers; see Chapter 2, p. 43.) Sets that can be put into one-to-one correspondence with the natural numbers are called *countable*, so the countable infinite sets are those with $\aleph_0$ elements. We have now seen that the integers are countable, and so also are all the fractions.

Are there sets which are *not* countable? Although we have extended the system, in passing from the natural numbers to first the integers and then the rational numbers, we have not actually increased the total number of objects that we have to work with. We have seen that the number of objects is actually countable in each case. Perhaps the reader has indeed got the impression by now that *all* infinite sets are countable. Not so; for the situation is very different in passing to the real numbers. It was one of Cantor's remarkable achievements to show that there are actually *more* real numbers than rationals. The argument that Cantor used is the 'diagonal slash' that was referred to in Chapter 2 and that Turing adapted in his argument to show that the halting problem for Turing machines is insoluble. Cantor's argument, like Turing's later one, proceeds by *reductio ad absurdum*. Suppose that the result we are trying to establish is false, i.e. that the set of all real numbers is countable. Then the real numbers between 0 and 1 are certainly countable, and we shall have *some* list providing a one-to-one pairing of all such numbers with the natural numbers, such as:

Natural numbers		Real numbers
0	$\longleftrightarrow$	0.10357627183. . .
1	$\longleftrightarrow$	0.14329806115. . .
2	$\longleftrightarrow$	0.02166095213. . .
3	$\longleftrightarrow$	0.43005357779. . .
4	$\longleftrightarrow$	0.92550489101. . .
5	$\longleftrightarrow$	0.59210343297. . .
6	$\longleftrightarrow$	0.63667910457. . .
7	$\longleftrightarrow$	0.87050074193. . .
8	$\longleftrightarrow$	0.04311737804. . .
9	$\longleftrightarrow$	0.78635081150. . .
10	$\longleftrightarrow$	0.40916738891. . .
.	.	.
.	.	.
.	.	.

I have marked out the diagonal digits in bold type. These digits are, for this particular listing,

$$1, 4, 1, 0, 0, 3, 1, 4, 8, 5, 1, \ldots$$

and the diagonal slash procedure is to construct a real number (between 0 and 1) whose decimal expansion (after the decimal point) differs from these digits in each corresponding place. For definiteness, let us say that the digit is to be 1 whenever the diagonal digit is different from 1 and it is 2 whenever the diagonal digit is 1. Thus, in this case we get the real number

$$0.21211121112\ldots$$

This real number cannot appear in our listing since it differs from the first number in the first decimal place (after the decimal point), from the second number in the second place, from the third number in the third place, etc. This is a contradiction because our list was supposed to contain *all* real numbers between 0 and 1. This contradiction establishes what we are trying to prove, namely that there is *no* one-to-one correspondence between the real numbers and the natural numbers and, accordingly, the number of real numbers is actually *greater* than the number of rational numbers and is *not* countable.

The number of real numbers is the infinite number labelled C. (C stands for *continuum*, another name for the system of real numbers.) One might ask why this number is not called $\aleph_1$, say. In fact the symbol $\aleph_1$ stands for the next infinite number greater than $\aleph_0$, and it is a famous unsolved problem to decide whether in fact $C = \aleph_1$, the so-called *continuum hypothesis*.

It may be remarked that the *computable* numbers, on the other hand, *are* countable. To count them we just list, in numerical order, those Turing machines which generate real numbers (i.e. which produce the successive digits of real numbers). We may wish to strike from the list any Turing machine which generates a real number that has already appeared earlier in the list. Since the Turing machines are countable, it must certainly be the case that the computable real numbers are countable. Why can we not use the diagonal slash on that list and produce a new computable number which is *not* in the list? The answer lies in the fact that we cannot computably decide, in general, whether or not a Turing machine should actually be in the list. For to do so would, in effect, involve our being able to solve the halting problem. Some Turing machines may start to produce the digits of a real number, and then get stuck and never again produce another digit (because it 'doesn't stop'). There is no computable means of deciding which Turing machines will get stuck in this way. This is basically the halting problem. Thus, while our diagonal procedure will produce some real number, that number will not be a computable number. In fact, this argument could have been used to *show* the existence of non-computable numbers. Turing's argument to show the existence of classes of problems which cannot be solved algorithmically, as was recounted in the last chapter, follows precisely this line of reasoning. We shall see other applications of the diagonal slash later.