SATAN, CANTOR, AND INFINITY

AND OTHER
MIND-BOGGLING PUZZLES

by

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open your envelope and find \$100. Then you know that the other envelope contains either \$50 or \$200. And so you gain \$100 if you do gain, and lose \$50, if you do lose. So isn't it obvious that the amount you stand to gain is greater than the amount you stand to lose? Isn't it obvious that \$100 is more than \$50? How can you possibly have any doubts on the matter?"

"You are looking at it the wrong way," insisted Alexander. "The amounts in the two envelopes are n dollars and 2n dollars for some n whose value we don't know. If you win on the trade, you will be moving up from n to 2n dollars, hence gaining n dollars, and if you lose on the trade, you will be moving down from 2n to n dollars, hence losing n dollars. Clearly the amounts are the same."

The two never did agree on this! Which do *you* believe to be the true one, Proposition 1 or Proposition 2?

17
OF TIME AND CHANGE

THE NEXT TIME our couple climbed the Sorcerer's tower, they found their host in a particularly philosophical mood. He had spent the morning reading early Greek and Chinese philosophy.

"Time is really a strange thing," he said. "Some mystics have claimed that time is unreal, and at times I am inclined to agree with them!"

"I see," said Annabelle. "At *times* you agree with them and at other times you don't, is that it?"

"That's about it," laughed the Sorcerer. "And speaking of time, I must read you a delightful passage I just found in a work by the ancient Chinese philosopher Chuangtse."

> There was a beginning. There was a time before that beginning. And there was a time before the time which was before that beginning. There was being. There was non-being. There was a time before that non-being. And there was a time before the time that was before that non-being. Suddenly there is being and there is non-being, but I don't know which of being and non-being is really being or really non-being. I have just said something, but

I don't know if what I have said really says something or says nothing.*

His two listeners had a good laugh—especially at the last line.

"I'm also reminded of Smullyan's recipe for immortality," said the Sorcerer. "Are you familiar with it?"

"No," said Alexander. "This I must hear!"

"It's really very simple: To be immortal, all you have to do are the following two things: (1) Always tell the truth; never make any false statements in the future. (2) Just say: 'I will repeat this sentence tomorrow!' If you do these two things, I guarantee that you will live forever!"

Of course, the Sorcerer was right: If today you truthfully say: "I will repeat this sentence tomorrow," then you will repeat the sentence tomorrow. Assuming you remain truthful tomorrow, then you will repeat the sentence the next day, then the next day, then the next day, . . .

"Theoretically, it's a perfect plan," said Annabelle, "but it's hardly the most *practical* plan in the world!"

"It reminds me of the White Knight's plan for getting over a gate," remarked Alexander.

"What plan is that?" asked the Sorcerer (who evidently had never read *Alice's Adventures in Wonderland* and *Through the Looking Glass*).

"As the White Knight explained it," said Alexander, "the only difficulty is with the feet: the *head* is high enough already. So you first put your head on the top of the gate—then the head's high enough; then you stand on your head—then the feet are high enough, you see; then you're over, you see."

"Very good," laughed the Sorcerer. "And concerning An-

*Wing-Tsit Chan, *A Source Book in Chinese Philosophy* (Princeton, N.J.: Princeton University Press, 1963), pp. 185, 186.

nabelle's objection to Smullyan's plan as not being practical, Smullyan presented his plan in a very amusing story of a man in search of immortality who visited a great sage in the East who was said to be an expert on this subject. The sage explained the plan of how to be immortal, but the man objected, as did Annabelle, on the grounds that the plan was not practical. He said to the sage: 'How can I truthfully say that I will repeat this sentence tomorrow when I don't know whether I'll even be alive tomorrow?' 'Oh,' replied the sage, 'you wanted a *practical* solution! No, I'm not very good at practice; I deal only in theory.' "

Annabelle and Alexander had a good laugh over that one.

"Speaking of immortality," said the Sorcerer, "I recall that when I was a lad, my uncle—who took a keen interest in all topics logical and philosophical—gave me a remarkable proof that it is logically impossible for anyone to die. Would you like to hear it?"

"Oh, yes!" they cried simultaneously.

"Well, my uncle put it this way: If a person dies, when does he die? Does he die while he is living or while he is dead? He can't die while he is dead, because once he is dead, he can no longer die—he is already dead. On the other hand, he can't die while he is still alive because he would then be dead and alive simultaneously, which is not possible. Therefore, he can't die at all."

"I would say," said Annabelle, after some thought, "that at the instant he dies, he is neither alive nor dead. The instant is the transition instant between life and death."

"That seems reasonable," said Alexander.

"No," said the Sorcerer, "you can't get out of it that easily! By 'dead' I mean no longer alive. In fact, to avoid any semantic confusion, I'll phrase the argument differently: at the instant of a person's death, is he alive or not alive at that instant?"

They thought some more about this.

"Is this related to Zeno's proofs of the impossibility of motion?" asked Annabelle.

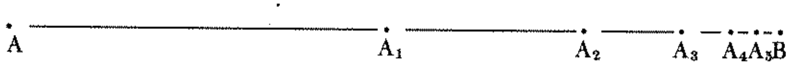
"There is some connection," replied the Sorcerer. "In fact, my uncle's argument can be generalized to give a new proof of the impossibility of motion."

"What are Zeno's arguments?" asked Alexander. "I have heard *of* them, but I have never heard them."

"I have read them," Annabelle said, "but I could never quite figure out where the fallacies lie. There *must* be fallacies, since things obviously *do* move. Just what are the fallacies?"

"Zeno had three proofs," replied the Sorcerer. "There is also a fourth proof, but this proof was poorly recorded and there does not seem to be uniform agreement as to just what that proof is. So I will confine myself to just the first three.

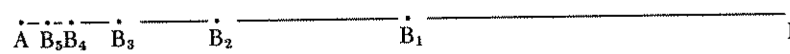
"The first argument is this: Suppose an object moves from a point A to a point B. Before it can get to B, it must get to the point A_1 midway between A and B; call this the *first* step. After completing the first step, the body must get from A_1 to the point A_2 midway between A_1 and B; call this the *second* step. Then it must take the third step—getting from A_2 to the point A_3 midway between A_2 and B, and so forth.



"After no finite number of steps will the body have reached B; that is, for each positive integer n, after it has performed the first n steps and is at a point A_n, it still has to take another step and go to the point A_{n+1} midway between A_n and B. So the body must take infinitely many steps to get to B and since this is impossible in a finite length of time, the object cannot move from A to B.

"Looking at it another way, before the object can get from A to B, it must first get to the midpoint B_1; but before it can get

to B_1, it must first get to the point B_2 midway between A and B_1; but before it can do *that*, it must get to the point B_3 midway between A and B_2, and so forth. Therefore, the body can't even get started!



"Zeno's second argument is about Achilles trying to overtake a tortoise. To begin with, let us say that Achilles is 100 yards behind the tortoise and that he is running ten times as fast as the tortoise. His first step is to run to the spot where the tortoise now is—that is, he runs 100 yards. When he reaches that spot, the tortoise will no longer be there; it will have moved 10 yards forward. Then Achilles takes the second step—he runs 10 yards to reach the spot where the tortoise was at the end of the first step, but when he gets there, the tortoise will have run (or rather walked) 1 yard further. Then when Achilles runs this yard forward, the tortoise will still be $1/10$ of a yard away, and so forth. In other words, whenever Achilles reaches the place where the tortoise had been, the tortoise is no longer there. Hence Achilles can never overtake the tortoise.

"Zeno's third proof strikes me as the most sophisticated; it is the proof about the flying arrow. Suppose an arrow is flying continuously forward during a certain time interval. Take any instant in that time interval. It is impossible that the arrow is moving *during that instant,* because an instant has duration *zero,* and the arrow cannot be at two different places at the same time. Therefore, at every instant the arrow is motionless, hence the arrow is motionless throughout the entire interval, which means that it can't move during the interval!"

"Yes, I remember those arguments, now that you have reminded me of them," said Annabelle, "and I am as much puzzled as ever. Surely, *something* must be wrong with them, since things obviously *do* move, but what is it that is wrong? Is there no logical explanation? Or is there something wrong with logic itself?"

"No, there is certainly nothing wrong with logic itself," laughed the Sorcerer, "and there certainly is a perfectly logical explanation as to just what the fallacies are in Zeno's arguments. I am really surprised that people are fooled by the first two arguments, though it is more understandable with the third, which is more difficult to see through."

"What *are* the fallacies?" asked Alexander.

"Well, let's start with the first one," said the Sorcerer. "In any argument that leads to a false conclusion, there must be a false step somewhere, hence there must be a *first* false step. So, where is the first false step in Zeno's first argument?"

The two of them thought about this for a while.

"I can't see any false step," said Alexander; "every step seems correct to me."

"To me also," said Annabelle.

"Really now!" said the Sorcerer. "Even the conclusion seems correct to you?"

"Of course not!" said Annabelle. "The conclusion is clearly false. Obviously things *do* move."

"Well then, if the conclusion is false and every step before the conclusion is correct, then obviously the first false step is the conclusion itself."

"I never thought of that!" said Annabelle.

"Most people don't, and that's the surprising thing!" said the Sorcerer. Of course, the conclusion is the first wrong statement! Every object must go through the infinite number of steps that Zeno described, but it is completely specious to conclude: 'Therefore the object can't move.' Where is the warrant for that inference? Zeno tacitly assumed that one cannot perform infinitely many steps in a finite length of time, and that assumption is totally unwarranted."

"That assumption seems perfectly reasonable to me!" said

Annabelle. "How can the sum of infinitely many quantities be finite?"

"That happens all the time in mathematics," replied the Sorcerer. "For example, the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots + \frac{1}{2^n} + \ldots$ all adds up to 2. Such an infinite series is called *convergent*; it converges to 2. On the other hand, the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots$ is what is called a *divergent* series; if you take enough terms of it, you'll get beyond a million; take enough more terms, you'll get beyond a trillion; whatever number you pick, no matter how large, that series eventually gets beyond that number. Such a divergent series is also said to approach infinity. However, the series relevant to Zeno's first problem is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, which converges to 1, and so Zeno was totally unjustified in concluding that the object could never get from A to B. A similar analysis applies to Zeno's second argument—in fact the series is $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots + \frac{1}{10^n} + \ldots$, which converges even more rapidly.

"The third argument of the flying arrow is more subtle, and I am afraid you don't know enough mathematics to understand the resolution. Those who have studied calculus know that at any instant during the interval, the arrow *is* in motion at that instant, and that Zeno was wrong in saying that if the arrow were in motion at an instant, then it would be in more than one place at the instant. Of course, the arrow is only at one place at any given instant, but that does *not* imply that the arrow is at rest at that instant."

"I don't understand," said Annabelle. "If an arrow in motion is only at one place at a given instant, how is it any different from an arrow at rest, which is also at only one place at an instant? How can one tell the difference between the two cases?"

"You ask a great and profound question," replied the Sorcerer, "and the purpose of the differential calculus—or at least one of its main purposes—is to answer just this sort of question. For the first time in the history of mankind, the inventors of the calculus—Newton and Leibniz—gave a perfectly precise definition of what it means for a body to be in motion *at an instant* and what it means to say that the velocity of the body is such and such *at an instant*. I cannot give you these definitions without first teaching you some fundamental notions of the calculus. For now, suffice it to say that the statement that a body has a certain velocity *at an instant* is really a statement not about just the instant but about successively smaller and smaller time intervals centered around the instant. I hope one day I can give you a more complete and technical account of the matter."

"You said before," said Alexander, "that your uncle's argument can be generalized to provide yet another proof of the impossibility of motion. How is that?"

"My uncle's argument can be generalized to show that nothing can ever change—which surely would have pleased the philosopher Parmenides! Suppose something changes from being in one state—call it state A—to being out of state A. At what instant can it make the change? It can't make the change when it is out of state A, because it is already out of state A. And it can't make the change while it is still in state A, because it would then be in state A and out of state A at the same time. And don't tell me that at the instant of change, it is neither in state A nor out of state A, because by *out of state A* I mean *not in state A*, and it is logically impossible for something to be neither in state A nor not in state A at the same time."

"How does that give another proof of the impossibility of motion?" asked Alexander.

"Well, obviously if an object moves away from a given position, it changes its state from being in that position to not being in that position. And so if changing from one state to another is impossible, then motion is impossible. . . .

"Yes, time and change are curious things! What really is time, anyway? As Augustine said: 'When asked what time is, I know not; when asked not, I know!'

"And speaking of time, I understand you will both be leaving us for a while?"

"Yes," replied Annabelle, "tomorrow we sail for home. My sister, Princess Gertrude, is getting married."

"How long will you be gone?" asked the Sorcerer.

"Probably a couple of months," replied Annabelle. "There are several things Alexander and I must attend to back home. But we definitely plan to come back. In fact, we are thinking of moving here. We are both absolutely intrigued by this whole subject of logic. No one at home seems to know much about it. You have no idea how grateful we are for all that you have taught us. But I am beginning to see that we have so much to learn! When we come back, could you tell us something about infinity? That's one subject that has always intrigued and puzzled me!"

"Ah, yes!" said the Sorcerer. "Infinity. That *is* a subject indeed! Yes, when you come back, I promise to give you a guided tour through infinity. Do come back soon, and bon voyage!"

18
WHAT IS INFINITY?

Our couple were away for more than two months. When they finally returned to the island, they lost no time in visiting the Sorcerer.

"Welcome back!" he said. "And so you want to know about infinity?"

"You have a good memory," said Annabelle.

"All right now," said the Sorcerer. "The first thing we should do is to carefully define our terms. Just what is meant by the word 'infinite'?"

"To me, it means endless," said Alexander.

"I would say the same," said Annabelle.

"That's not quite satisfactory," said the Sorcerer. "A circle is endless, in the sense of having no beginning or end, and yet you wouldn't say it's infinite—that is, it has only finite length, though it does have infinitely many points.

"I want to talk about infinity in the precise sense used by mathematicians. Of course, there are other uses of the word; for example, theologians often refer to God as infinite, though some of them are honest enough to admit that when applied to God the word has a different meaning than when applied to anything else. Now, I do not wish to disparage the theological or any other nonmathematical use of the word, but I want to make it clear that what I'm proposing to discuss is infinity in the purely *mathematical* sense of the term. And for this we need a precise definition.

"The word 'infinite' is obviously an adjective, and the first thing we must agree upon is the sort of things to which the adjective is applicable. What sort of things can be classified as either finite or infinite? Well, in the mathematical use of the term, the answer is that it is *sets*, or collections of things, that can be called *finite* or *infinite*. We say that a *set* of objects has finitely many or infinitely many members, and now we must make these notions precise.

"The clue here lies in the notion of a one-to-one correspondence between one set and another. For example, a flock of seven sheep and a grove of seven trees are related to each other in a way that neither is related to a pile of five stones, for the set of seven sheep can be paired with the set of seven trees, for example, by tethering each sheep to a tree, so that each sheep and each tree belongs to exactly one of the pairs. Or, in mathematical terms, a set of seven sheep can be put into a 1 to 1 correspondence with a set of seven trees.

"Another example: Suppose you look into a theater and see that every seat is taken and that no one is standing—and also, that no one is sitting on anyone's lap; there is one and only one person to each seat. Then, without having to count the number of people, or the number of seats, you know that the numbers are the same, because the set of people is in a 1 to 1 correspondence with the set of seats: each person corresponds to the seat he is occupying.

"Now, I know that you are familiar with the set of natural numbers, though you might not be familiar with them by that name. The natural numbers are simply the numbers 0, 1, 2, 3, 4, . . . That is, by a natural number is meant either zero or any positive whole number."

"Is there such a thing as an *unnatural* number?" asked Annabelle.

"No, I have never heard of that," said the Sorcerer, "and I must say the idea strikes me as funny! Anyway, from now on I will be using the word *number* to mean *natural number*, unless I say something to the contrary.

"Now, given a natural number n, just what does it mean to say that a certain set has exactly n members? For example, what does it mean to say that there are exactly five fingers on my right hand? It means that I can put the set of fingers on my right hand into a 1 to 1 correspondence with the set of positive whole numbers from 1 to 5—say, by letting my thumb correspond to 1, the next finger to 2, the middle finger to 3, the next finger to 4, and the little finger to 5. And, in general, given any positive whole number n, we say that a set has (exactly) n members if the set can be put into a 1 to 1 correspondence with the set of positive integers from 1 to n. A set with n elements is also called an *n − element set*. And the process of putting an n − element set into a 1 to 1 correspondence with the set of positive integers has a popular name—the popular name of this process is *counting*. Yes, that's exactly what counting is.

"And so, I have told you what it means for a set to have n elements, where n is a *positive* whole number. What about when n is zero; what does it mean for a set to have 0 elements? It obviously means that the set has no elements at all."

"There are such sets?" asked Alexander.

"There is only one such set," replied the Sorcerer. "This set is technically called the *empty set*, and is extremely useful to mathematicians. Without it, exceptions would constantly have to be made and things would get very cumbersome. We want, for example, to be able to speak about the set of people in a theater at a given instant. It may happen that there are no people there at a given instant, in which case we say that the set of people then in the theater is empty—just as we speak of an empty theater. This is not to be confused with no theater at all! The theater still exists as a theater; there just happen to be no people in it. Likewise, the empty set exists as a set, but it has no members.

"I recall a charming incident: Many years ago, I told a lovely lady musician about the empty set. She seemed surprised and said: 'Mathematicians really use this notion?' I replied: 'They certainly do!' She asked: 'Where?' I replied: 'It's used all over the place.' She thought for a moment and said: 'Oh, yes. I guess it's like the rests in music.' I thought that was quite a good analogy!

"Smullyan relates an amusing incident. When he was a graduate student at Princeton University, one of the famous mathematicians there said during a lecture that he hated the empty set. At the next lecture, he made use of the empty set. Smullyan raised his hand and said, 'I thought you said that you disliked the empty set.' The professor replied: 'I said I disliked the empty set. I never said that I don't *use* the empty set!' "

"You haven't yet told us," said Annabelle, "what you mean by 'finite' and 'infinite.' Aren't you going to?"

"I was just about to," replied the Sorcerer. "All the things I've said so far are just leading up to the definition. We say that a set is *finite* if there exists a natural number n such that the set has exactly n elements—which we recall means that the set can be put into a 1 to 1 correspondence with the positive integers from 1 to n. If there is no such natural number n, then the set is called infinite. It's as simple as that. Thus a 0-element set is finite; a 1-element set is finite, a 2-elements set is finite, . . . and an n-element set is finite, where n is any natural number. But if for every natural number n, it is false that the set has exactly n elements, then the set is infinite. Thus, if a set is infinite, then for any natural number n, if we remove n elements from the set, there will be elements left over—in fact, infinitely many elements will be left over.

"Do you see why this is true? Let's first consider a simple problem. Suppose I remove a single element from an infinite set. Is what remains necessarily infinite?"

"It would certainly seem so!" said Annabelle.

"It certainly does!" said Alexander.

"Well, you are right, but can you prove it?"

The two thought about it, but had difficulty proving it. It seemed too obvious to need proof. But it is not difficult to prove this from the very definitions of "finite" and "infinite." These definitions must be used.

• 1 •

How is this proved?

It took a bit of prodding, but the two finally came up with a proof that satisfied the Sorcerer.

Hilbert's Hotel. "Infinite sets," said the Sorcerer, "have some curious properties that at times have been labeled paradoxical. They are not really paradoxical, but they are a bit startling when first encountered. This is well illustrated by the famous story of Hilbert's Hotel.

"Suppose we have an ordinary hotel with only a finite number of rooms—say a hundred. Suppose all the rooms are taken and each room has one occupant. A new person arrives and wants a room for the night, but neither he nor any of the hundred guests is willing to share a room. Then, it is impossible to accommodate the new arrival; one cannot put 101 people into a 1 to 1 correspondence with 100 rooms. But with infinite hotels (if you can imagine such a thing) the situation is different. Hilbert's Hotel has infinitely many rooms—one for each positive integer. The rooms are numbered consecutively—Room 1, Room 2, Room 3, . . . Room n, . . . and so forth. We can imagine the rooms of the hotel arranged linearly—they start at some definite point and go infinitely far to the right. There is a first room, but there is no last room! It's important to remember that there is no last room—just as there is no last natural number. Now again, we assume that all the rooms are occupied—each room has one guest. A new person arrives on the scene and wants a room. The interesting thing is that it *is* now possible to accommodate him. Neither he nor any of the other guests is willing to share a room, but the guests are all cooperative in that they are willing to change their rooms, if requested to."

• 2 •

How can this be done?

"Now for another problem," said the Sorcerer, after the solution to this last problem had been discussed. "We consider the same hotel as before. But now, instead of just one new person arriving, an infinite number of new guests arrive—one for each positive integer n. Let us call the old guests P_1, P_2, . . . P_n, . . . , and the new arrivals Q_1, Q_2, . . . , Q_n, . . . The new Q-people all want accommodations. The surprising thing is that it can be done!

• 3 •

How?

"Now for a still more interesting problem," said the Sorcerer. "This time we have infinitely many hotels—one for each positive integer n. The hotels are numbered Hotel 1, Hotel 2, . . . Hotel n, . . . and each has infinitely many rooms—one for each positive integer. The hotels are arranged in a rectangular square array—thus:

Hotel 1	1	2	3	4	5	6	7 . . .
Hotel 2	1	2	3	4	5	6	7 . . .
Hotel 3	1	2	3	4	5	6	7 . . .
.	.						
.	.						
.	.						
.	.						
Hotel *n*	1	2	3	4	5	6	7 . . .
.	.						
.	.						
.	.						
.	.						

"The whole chain of hotels is under one management. All the rooms of all the hotels are occupied. One day the management decides to shut down all the hotels but one, in order to save energy. But this means shifting all the inhabitants of all the hotels into just one of the hotels—again, only one person to a room."

• 4 •

Is this possible?

"You see what these problems reveal," said the Sorcerer. "They show that an infinite set can have the strange property of being able to be put into a 1 to 1 correspondence with a proper part of itself. Let me make this more precise.

"A set A is called a *subset* of a set B if every element of A is also an element of B. For example, if A is the set of numbers from 1 to 100 and B is the set of numbers from 1 to 200, then A is a subset of B. Or if E is the set of all even integers and N is the set of all integers, then E is a subset of N. A subset A of B is called a *proper* subset of B if A is a subset of B but does not contain *all* the elements of B. In other words, A is a proper subset of B if A is a subset of B, but B is not a subset of A. Now, let P be the set $\{1, 2, 3, \ldots n, \ldots\}$ of all positive integers and let $P-$ be the set $\{2, 3, \ldots n, \ldots\}$ of all positive integers other than 1. We have seen in the first problem of Hilbert's Hotel that that P can be put into a 1 to 1 correspondence with $P-$, yet $P-$ is a *proper* subset of P! Yes, an infinite set can have the strange property of being able to be put into a 1 to 1 correspondence with a *proper* subset of itself! This was known long ago. In 1638, Galileo pointed out that the *squares of the positive integers* can be put in a 1 to 1 correspondence with the positive integers themselves, thus

$$\begin{array}{cccccccc} 1, & 4, & 9, & 16, & 25, & \ldots & n^2, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1, & 2, & 3, & 4, & 5, & \ldots, & n, & \ldots \end{array}$$

"This seemed to contradict the ancient axiom that the whole is greater than any of its parts."

"Well, doesn't it?" asked Alexander.

"Not really," replied the Sorcerer. "Suppose that A is a proper subset of B. Then, in one sense of the word 'greater,' B is greater than A—namely, in the sense that B contains all elements that A contains and some elements that B doesn't contain. But that does not mean that B is *numerically* greater than A."

"I'm not sure I know what you mean by numerically greater," said Annabelle.

"Good point!" said the Sorcerer. "First of all, what do you think it means for one set A to be *of the same size* as another set B?"

"I guess it means that A can be put into a 1 to 1 correspondence with B," said Annabelle.

"Right! And what would you guess it means to say that A is of *smaller* size than B, or that, numerically, A has fewer elements than B?"

"I guess it would mean that A can be put into a 1 to 1 correspondence with a proper subset of B."

"Nice try," said the Sorcerer, "but it won't work. That definition would be fine for finite sets, but not for infinite sets. The trouble is that it might be possible for A to be put into a 1 to 1 correspondence with a proper subset of B, and it might also be possible for B to be put into a 1 to 1 correspondence with a proper subset of A. In which case would you want to say that each is of smaller size than the other? For example, let O be the set of odd positive integers and E be the set of even positive integers. Obviously, O can be put into a 1 to 1 correspondence with E.

$$\begin{array}{cccccc} 1, & 3, & 5, & 7, & 9, & \ldots 2n-1 \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2, & 4, & 6, & 8, & 10, & \ldots 2n \ldots \end{array}$$

"But also O can be put into a 1 to 1 correspondence with a *proper* subset of E thus

$$\begin{array}{cccccc} 1, & 3, & 5, & 7, & 9, & \ldots 2n-1 \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 4, & 6, & 8, & 10, & 12, & \ldots 2n+2 \end{array}$$

"At the same time, E can be put into a 1 to 1 correspondence with a proper subset of O thus

$$\begin{array}{cccccc} 2, & 4, & 6, & 8, & 10, & \ldots 2n \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 3, & 5, & 7, & 9, & 11, & \ldots 2n+1 \ldots \end{array}$$

"Now, you certainly wouldn't want to say that E and O are of the same size, yet E is of smaller size than O and also O is of smaller size than E! No, that definition won't serve."

"Then what is the correct definition of 'smaller than' when applied to infinite sets?" asked Annabelle.

"The correct definition is this: We say that A is *smaller in size than B*, or that B is *greater in size than A* if the following two conditions are met: (1) A can be put into a 1 to 1 correspondence with a proper subset of B; (2) A *cannot* be put into a 1 to 1 correspondence with all of B.

"It is crucial that *both* conditions be met," emphasized the Sorcerer, "in order that it can be correctly said that A is smaller than B. To say that A is smaller than B means first of all that A can be put into a 1 to 1 correspondence with a subset of B and also that *every* 1 to 1 correspondence between A and a subset of B must leave out some elements of B.

"And now comes a fundamental question," said the Sorcerer. "Is it the case that any two infinite sets are necessarily of the same size, or do infinite sets come in different sizes? That is the first question to be answered in building up a theory of infinity, and fortunately the question was answered by Georg Cantor towards the end of the last century. The answer created a storm and started a whole new branch of mathematics whose ramifications are fantastic!

"I will give you Cantor's answer at our next session. Meanwhile, do you have a guess as to what the answer is? Are all infinite sets of the same size, or do they come in different sizes?"

Remark. I raise this problem to my students in my beginning logic classes, and among those students who don't know the answer, about half guess that all infinite sets are of the same size and about half guess otherwise.

Those of you who don't already know the answer, would you care to hazard a guess before turning to the next chapter?

Solutions

1. Let us first show that when you add an element to a finite set, you get a finite set. Well, suppose a set A is finite. By definition, this means that for some natural number n, the set A has n elements. If we add a new element to A, the resulting

set then obviously has n + 1 elements, hence by definition is finite.

From this, it immediately follows that if we remove an element from an infinite set B, the resulting set must be infinite, for if it were finite, we could put back the element and the resulting set—which is the original set B that we started with—would be finite, which it isn't.

2. All that the management need do is to request that each inhabitant move one room to the right—in other words, the occupant of Room 1 goes to Room 2, the occupant of Room 2 goes to Room 3, . . . the occupant of Room n moves to Room n + 1. Since the hotel has no last room (unlike the more normal finite hotel), no one will be out in the cold. (In a finite hotel, the person in the last room would have no room to his right). After all the guests have been kind enough to move, Room 1 is now empty and the newly arrived guest can take it.

Mathematically, what we have done is to put the set of all positive integers into a 1 to 1 correspondence with the set of all positive integers from 2 on. Of course, the manager of the hotel could have done a similar thing if a hundred million new guests had arrived instead of just one. The manager would then have asked each guest to move a hundred million rooms to the right (the person in Room 1 would move to Room 100,000,001; the person in Room 2 would move to Room 100,000,002, and so forth). For any natural number n, the hotel could accommodate n new guests—just have each guest move n rooms to the right, leaving the first n rooms empty for the new guests.

3. Now, if infinitely many new guests $Q_1, Q_2, \ldots Q_n, \ldots$ arrive, the solution is a little different. One false solution that has been suggested is that the manager first asks every one of the old guests to move one room to the right. Then, one new guest is put into the empty Room 1. Then the manager again asks everyone to move one room to the right, and so Room 1 is

again vacant and a second new guest is put in Room 1. Then this operation is repeated, again and again, infinitely many times, and so sooner or later every new guest gets into the hotel.

But what a horribly restless solution! No one keeps a room permanently and in no finite time would all the guests be settled—infinitely many shiftings would be required. No, the whole thing can be done cleanly with only one shifting. Can you see what this shifting is?

The shifting is that each old guest doubles his room number—that is, the person in Room 1 goes to Room 2, the one in Room 2 goes to Room 4, the one in Room 3 goes to Room 6, . . . the one in Room n goes to Room 2n. All these are, of course, done simultaneously, and after the shifting, all the even-numbered rooms are occupied and all the infinitely many odd-numbered rooms are now free. And so, the first new guest Q_1 goes to the first odd-numbered room—Room 1; Q_2 goes to Room 3; Q_3 goes to Room 5, and so forth (Q_n moves to Room $2n-1$).

4. We first "number" all the occupants of all the rooms of all the hotels according to the following plan:

1		4		9		16	.	.	.
↓		↑		↑		↑	.	.	.
2	→	3		8		15	.	.	.
				↑		↑	.	.	.
5	→	6	→	7		14	.	.	.
						↑	.	.	.
10	→	11	→	12	→	13	.	.	.
.		.		.		.	.	.	.
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And so, each of the guests is "tagged" with a positive integer. Then all of them vacate their rooms and wait outside for a bit. Then the management shuts down all the hotels but one, and each guest is asked to occupy the room whose number is the same as the number with which he is tagged—the person tagged n goes to Room n.

19

CANTOR'S FUNDAMENTAL DISCOVERY

"WELL," SAID THE Sorcerer at the next session, "have you thought about the matter? Have you any guess as to whether there is more than one infinity or only one?"

One of the two (I forget which) guessed one way, and the other, the other.

"The curious thing," said the Sorcerer, "is that Cantor originally conjectured that any two infinite sets must be of the same size, and as I understand it, he spent twelve years trying to prove his conjecture. Then, in the thirteenth year, he found a counterexample—which I like to call a 'Cantor-example.' Yes, there is more than one infinity—there are infinitely many, in fact. And this basic discovery is due to Cantor.

"Now, here is what Cantor did. A set is called *denumerably infinite*, or, more briefly, *denumerable*, if it can be put into a 1 to 1 correspondence with the set of positive integers. And so the question considered by Cantor is: Is every infinite set denu-

Q: merable? As I said, he first guessed that every infinite set is denumerable, and only later discovered the real truth. What he did, during his initial investigation, was to consider sets that appeared to be too large to be denumerable, but then by some clever device he was able to enumerate them after all."

"What do you mean by *enumerating* a set?" asked Annabelle.

"To *enumerate* a set means to put it into a 1 to 1 correspondence with the set of positive integers. In fact, the word 'enumerable' is used synonymously with 'denumerable.' Anyway, as I said, Cantor considered one set after another which at first sight appeared to be non-denumerable—which means infinite but not denumerable—and found a clever way to enumerate it.

"To illustrate his method, let us imagine that I am the Devil and you are my victims in Hell. That's not too hard to imagine, is it?"

Annabelle and Alexander had a good laugh at that thought.

"Now, I tell you that I will give you a test. I tell you: 'I am thinking of a positive whole number, which I have written down on this folded piece of paper. Each day, you have one and only one guess as to what the number is. If and when you correctly guess it, you can go free, but not before then.' Now, is there some strategy by which you can be sure of getting out sooner or later?"

"Of course," said Alexander. "On the first day I ask if it's 1, on the second day, if it's 2, and so forth. Sooner or later I'm bound to hit your number."

"Right," said the Sorcerer. "Now, my second test is a tiny bit more difficult. This time, I write down either a positive whole number or a negative whole number—I write either one of the numbers 1, 2, 3, 4, . . . or one of the numbers -1, -2, -3, -4, . . . and again each day you have one and only one guess as to what the number is. Now, do you have a strategy that will surely get you out sooner or later?"

"Of course," said Annabelle. "On the first day I ask whether the number is $+1$, on the next day, I ask if it is -1, then I keep going $+2$, -2, $+3$, -3, $+4$, -4, . . . and so on. Sooner or later I must hit your number."

"Right," said the Sorcerer, "and you see what this means.

On the surface, it would seem that the set of positive and negative whole numbers taken together should be larger than—in fact twice the size of—the positive whole numbers alone. Yet, you have just seen how to put the set of positive and negative whole numbers into a 1 to 1 correspondence with the set of positive whole numbers, and so the two sets are really of the same size after all. The set of all positive and negative whole numbers taken together is denumerable. The problem you have just solved is substantially the same as the second problem I gave you concerning Hilbert's Hotel. As you recall, there were denumerably many people in denumerable many rooms, and then a second denumerable set of people came along and we wanted to put those two sets together and accommodate all the members.

"The next test I give my victim is definitely more difficult. This time, I write down *two* numbers on a piece of paper, or maybe the same number written twice. For example, I might write 3, 57, or I might write 17, 206, or I might write 23, 23. Each day, you have one and only one guess as to what the two numbers are. You are not allowed to guess one of them one day and the other another day; you must guess them both on the same day. Now, do you believe there is a strategy by which you can certainly get out sooner or later?"

"I doubt it," said Annabelle. "There are infinitely many possibilities for the first number that you write down, and with each of those possibilities there are infinitely many possibilities for the second number, and so it would seem that an infinity times itself should be bigger than the infinity you start with."

"So it might *seem*," said the Sorcerer, "and so it did seem to many at Cantor's time, but appearance is sometimes deceptive. Yes, there *is* a strategy for certainly getting out. The set of possibilities that you are dealing with is really denumerable after all. Can you find the strategy?"

"Amazing!" said Annabelle, and Alexander agreed. The two then put their heads together and came up with a simple strategy that definitely works.

• 1 •

What strategy will work?

"Suppose I make the problem slightly harder by now requiring not only that you guess what the two numbers are, but also in what order they are written—say, one is written to the left of the other. Can you now get out for sure?"

"Of course," said Annabelle. "This problem is easy, now that the last problem was solved."

• 2 •

What is the solution?

"Then let me ask you this," said the Sorcerer. "What about the set of all positive fractions? Is that set denumerable or not? You are now in a good position to answer that question. By a *positive fraction* is simply meant a quotient of two positive integers—numbers like $3/7$ or $21/13$."

• 3 •

Is the set of positive fractions denumerable?

"The answer was quite a shock to many mathematicians of Cantor's time," said the Sorcerer. "And now I have a more difficult problem for you. This time, I write down some *finite* set of positive integers. I'm not telling you how many numbers are in the set, nor what the highest number of the set is. Each

day, you have one and only one guess as to what the set is. If and when you correctly guess the set, you go free. Now, do you think there is a strategy for going free?"

The two felt that it was quite unlikely.

"There *is* such a strategy," said the Sorcerer. "The set of all *finite* sets of positive integers is denumerable."

• 4 •

How can the set of all finite sets of positive integers be enumerated? What strategy would you use to get free?

"What about the set of *all* sets of positive integers—all infinite sets as well as all finite ones?" asked Annabelle. "Is that set denumerable or not? Or is the answer unknown?"

"Ah!" said the Sorcerer. "That set is non-denumerable—that is Cantor's great discovery!"

"No one has yet found a way of enumerating that set?" asked Alexander.

"No one has, and no one ever will, because it is logically impossible to enumerate that set."

"How is that known?" asked Annabelle.

"Well, let's first look at it this way: Imagine a book with denumerably many pages—page 1, page 2, page 3, . . . page n, . . . On each page is written down a description of a set of positive integers. You own the book. If *every* set of positive integers is listed in the book, you win a grand prize. But I tell you that you can't win the prize because I can describe a set of positive integers that couldn't possibly be described on any page of the book."

• 5 •

Describe a set of positive integers that is definitely not described on any page of the book.

"And so you see," said the Sorcerer, after explaining the solution of the last problem, "that the set of all sets of positive integers is non-denumerable. It is larger than the set of positive integers."

"You haven't shown that," said Annabelle. "You have shown that the set of all sets of positive integers—is there a name for this set?"

"Yes," said the Sorcerer. "For any set A, the set of all subsets of A is called the *power set of A* and is denoted P(A). We can use N for the set of positive integers, and so the set of all subsets of N—which is the set of all sets of positive integers—is thus the power set of N and is denoted P(N).

"All right," said Annabelle. "Anyway, you have indeed shown that P(N) is non-denumerable—that P(N) cannot be put into a 1 to 1 correspondence with N, and of course P(N) is infinite, but to conclude that P(N) is therefore *larger* than N is unwarranted, since you haven't shown that N can be put into a 1 to 1 correspondence with some *subset* of P(N). Don't you have to do that in order to complete your argument?"

"We have already put N into a 1 to 1 correspondence with a subset of P(N)," replied the Sorcerer.

• 6 •

When was that done?

After the Sorcerer reminded Annabelle of Problem 4, one of the two (I forget whether it was Annabelle or Alexander) raised a question: "We have seen that the set of all *finite* sets of positive integers is denumerable; hence it can be enumerated in some infinite sequence $S_1, S_2, \ldots S_n, \ldots$ Why can't we apply Cantor's argument and get a set S different from all the sets $S_1, S_2, \ldots S_n, \ldots$? Doesn't this create a paradox?"

• 7 •

Does it create a paradox?

"I have a question," said Alexander after the last matter had been settled. "We know that the set of all *finite* sets of positive integers is denumerable. What about the set of all *infinite* sets of positive integers? Is that set denumerable or not?"

• 8 •

What is the answer to Alexander's question?

"Another question. We have seen that P(N) is larger than N. Is there a set larger than P(N)?" asked Annabelle.

"Why certainly," replied the Sorcerer. "The fact that P(N) is larger than N is only a special case of Cantor's theorem, which is:

Theorem C (Cantor's theorem). For *any* set A, the set P(A) of all subsets of A is larger than A.

"The proof of Cantor's theorem," said the Sorcerer, "is not substantially different from the proof I have given you for the special case when A is the set N of positive integers. The idea behind the proof has been nicely illustrated by Smullyan in the following problem: In a certain universe, every set of inhabitants forms a club. The registrar of the universe would like to name each club after an inhabitant in such a way that no two clubs are named after the same inhabitant and each inhabitant has a club named after him. It is not necessary that an inhabitant be a member of the club named after him. Now, for a universe with only finitely many inhabitants, this is clearly impossible, for there are then more clubs than inhabitants (if n is the number of inhabitants, then 2^n is the number of clubs). But this particular universe has infinitely many inhabitants, and so the registrar sees no reason why this should not be possible. However, every scheme he has ever tried has failed—there are always clubs left over. Why is the registrar's scheme impossible to carry out?"

• 9 •

Explain why the registrar's scheme is impossible and how this is related to Cantor's theorem.

"As a consequence of Cantor's theorem," said the Sorcerer after Annabelle and Alexander understood the proof, "there must be an infinite number of different-size infinities, for we can start with N, the set of positive integers, then we have its power set P(N)—the set of all subsets of N—which is larger than N, but then again by Cantor's theorem the power set of *this* new set—in other words P(P(N))—is larger than P(N), and then the set P(P(P(N))) is larger still, and so forth. Thus, for *any* set, there is a larger set, and so the sizes of sets are endless."

Solutions

1. For each n, there are only finitely many possibilities for a pair whose highest number is n—there are exactly n such possibilities in fact. Thus there is only one possibility for the pair whose highest number is 1—namely (1, 1). There are two possibilities for the pair whose highest number is 2—namely (1, 2) and (2, 2). Then, there are the three possibilities for the pair whose highest number is 3—(1, 3), (2, 3), and (3, 3), and so forth. And so we enumerate them in the order (1, 1), (1, 2), (2, 2), (1, 3), (2, 3), (3, 3), (1, 4), (2, 4), (3, 4), (4, 4), . . . (1, n), (2, n), . . . (n − 1, n), (n, n), . . .

2. In this case, it might take us roughly twice as long to get out, but we still get out sooner or later by enumerating the ordered pairs in the order (1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (2, 3), (3, 3), (3, 2), (3, 1), . . . (1, n), (2, n) . . . (n − 1, n), (n, n), (n, n − 1), . . . (n, 2), (n, 1), (1, n + 1), . . .

3. This is really the same problem as the last, except that we have one integer *above* the other (as the numerator) instead of to the left of the other. Thus, we can enumerate the positive fractions in the order $1/1$, $1/2$, $2/2$, $1/3$, $2/3$, $3/3$, $3/2$, $3/1$, $1/4$, $2/4$, $3/4$, $4/4$, $4/3$, $4/2$, $4/1$, . . . Of course, we could get out a bit sooner by not naming duplications, such as $2/2$ (which is really $1/1$), and $3/3$, and $2/4$ (which duplicates $1/2$), and so forth.

4. A set whose members are elements $a_1, a_2, \dots a_n$ is written $\{a_1, a_2, \dots a_n\}$. Now, there is only one set whose highest number is 1, namely {1}. There are two sets whose highest number is 2, namely {1, 2} and {2}. There are four sets whose highest number is 3, namely {3}, {1, 3}, {2, 3}, {1, 2, 3}. In general, for any positive integer n, there are 2^{n-1} sets whose highest number is n. The reason is this: For any n, there are 2^n subsets of the set of positive integers from 1 to n (and this includes the empty set!). And so any set whose highest number is n consists of n together with some subset of the integers from 1 to n − 1, and there are 2^{n-1} such subsets.

Anyhow, the important thing is that for each n, there are only *finitely* many sets of positive integers whose highest member is n. And so, I first name the empty set. Then, I name the set whose highest number is 1. Then I go through the sets whose highest number is 2 (the order doesn't really matter), then the sets whose highest number is 3, and so forth.

5. Given any number n, either n belongs to the set listed on page n or it doesn't. We let S be the set of all numbers n such that n does *not* belong to the set listed on page n. For each n we let S_n be the set listed on page n. Our definition of S is such that for every n, the set S must be different from S_n, because n belongs to S if and only if n *doesn't* belong to S_n. This means that either n is in S but not in S_n, or n is not in S but in S_n. In either case, S must be different from S_n, because one of those two sets contains n and the other doesn't.

To give a more concrete idea of the construction of the set S, suppose the sets listed on the first ten pages are as follows:

Page 1—Set of all even numbers
Page 2—Set of all (positive whole) numbers
Page 3—The empty set
Page 4—Set of all numbers greater than 100
Page 5—Set of all numbers less than 58
Page 6—Set of all prime numbers
Page 7—Set of all numbers that are not prime
Page 8—Set of all numbers divisible by 4
Page 9—Set of all numbers divisible by 7
Page 10—Set of all numbers divisible by 5

I have just listed the first ten sets at random. Now, what will S look like as far as the first ten numbers are concerned? Well, what about 1; should S contain 1? Is 1 a member of the set listed on page 1; that is, is 1 an even number? No, it is not, so 1 doesn't belong to S_1; so we put 1 in S. What about 2? Well, 2 certainly is in S_2 (2 is a positive whole number), and so we *don't* allow 2 in S. The number 3 is certainly not in S_3 (no number is in the empty set), and so we take 3 as a member of S. Now, 4 is not in S_4 (4 is not greater than 100), so 4 is in S. We let the reader check the next six cases: 5 is in S, since 5 is not in S_5; 6 is not in S_6 (6 isn't prime), so 6 is in S; 7 is not in S_7, so 7 is in S, 8 is in S_8, so 8 is not in S; 9 isn't in S_9, so 9 gets put in S; 10 is in S_{10} (10 is divisible by 5), so 10 is left out of S. Thus in listing the entries of S, let us put the number n in the n^{th} place if n is in S, and let us put a blank in the n^{th} place to signify that n is definitely left out of S. Then, the first ten places of our list look like this: 1, —, 3, 4, 5, 6, 7, —, 9, — . . . . We now see that our set S is different from S_1 since it contains 1 and S_1 doesn't. Also, S is different from S_2 because it *doesn't* contain 2 and S_2 does. And so you see, for each n, either S contains n and S_n doesn't, or S doesn't contain n and S_n does, so S cannot equal S_n. Thus, the set S is different from every one of the sets listed in the book.

Of course, we don't really need the book for this argument to hold true. The whole point is that given *any* enumeration $S_1, S_2, \ldots S_n, \ldots$ of sets of positive integers, there exists a set S of positive integers that is different from every one of the S_n's—namely, the set of all n such that n doesn't belong to S_n. Thus, the infinite sequence $S_1, S_2, \ldots S_n, \ldots$ fails to catch *every* set of positive integers, since the set S has been left out. And so, the set of all sets of positive integers is not denumerable.

6. In Problem 4 we showed that the set of all *finite* subsets of N is denumerable, and thus N can be put into a 1 to 1 correspondence with the set of all finite subsets of N. Obviously, the set F of all finite subsets of N is a subset of the set of *all* subsets of N—thus F is a subset of P(N) and N can be put into a 1 to 1 correspondence with F.

7. Of course not! The set S is indeed different from each of the finite sets S_n, but all that means is that the set S is not finite.

8. We know that the set of all finite sets of positive integers is denumerable; hence it can be enumerated in some infinite sequence $F_1, F_2, \ldots F_n. \ldots$ That is, to each n we can correspond some finite set F_n of positive integers, and the correspondence is such that every finite set of positive integers is some F_n or other. Now, suppose that the set of all *infinite* sets of positive integers were denumerable. Then it could be enumerated in some infinite sequence $I_1, I_2, \ldots I_n, \ldots$ where for each n, I_n is the infinite set corresponding to the integer n. But then, we could enumerate *all* sets of integers—finite and infinite—in the order $F_1, I_1, F_2, I_2, \ldots F_n, I_n, \ldots$ which is contrary to the fact that the set of all sets of positive integers is non-denumerable.

9. Suppose the registrar's scheme could be carried out. Then we get a contradiction as follows: Call an inhabitant *sociable* if he belongs to the club named after him and *unsociable* if he doesn't. Since in this universe every set of inhabitants constitutes a club, then the set of all unsociable inhabitants constitutes a club. This club is named after someone—say *John.* Thus, every member of John's Club is unsociable and every unsociable inhabitant belongs to John's Club. Is John sociable or not? Either way, we get a contradiction: Suppose John is sociable. This means that John belongs to John's Club, but only unsociable people belong to John's Club, so this is out. On the other hand, suppose that John is unsociable. Since every unsociable inhabitant belongs to John's Club, then John, being unsociable, would have to belong to John's Club, which makes John sociable. And so whether John is sociable or not, we get a contradiction. Thus, the registrar's scheme cannot be carried out.

The relation of this problem to Cantor's theorem should be obvious—it is only a special case of Cantor's theorem. Instead of a universe of people, consider an arbitrary set A. Suppose we have a 1 to 1 correspondence between A and the set P(A) of all subsets of A. We get the following contradiction: Each element x of A corresponds to one and only one subset of A, which we might call x's set. Now, let S be the set of all elements x of A such that x does *not* belong to x's set. (In application to the problem above, S is the set of unsociable inhabitants.) Some element b of A corresponds to this set S, and so b's set is the set of all x's having the property that x doesn't belong to x's set. If b belongs to b's set, then b is one of the elements having the property of not belonging to b's set, which is a contradiction. If b doesn't belong to b's set, then b has the property of not belonging to b's set; but every element having this property belongs to b's set, and so b must then belong to b's set, and we again have a contradiction. This proves that there exists no 1 to 1 correspondence between A and its power set P(A).

Of course, A can be put into a 1 to 1 correspondence with some *subset* of P(A) as follows: For any element x, by $\{x\}$ is meant the set whose only element is x. (Such a set $\{x\}$ is called a *unit* set or a *singleton.* It has only one element, regardless of how many elements x itself may have.) Well then, we can let each element x of A correspond to the singleton $\{x\}$. This correspondence is obviously 1 to 1, and of course $\{x\}$ is a subset of A (since every element of $\{x\}$, of which there is only one—x itself—is an element of A). And so A is thus in a 1 to 1 correspondence with a set consisting of *some* of the elements of P(A).

Since A can be put into a 1 to 1 correspondence with a subset of P(A) and A cannot be put into a 1 to 1 correspondence with all of P(A) (as we have shown), then by definition, P(A) is larger than A. This proves Cantor's theorem.