CHAPTER

2 Basic Concepts and Proofs

People encounter the geometric concepts of perpendicularity, complementary angles, and supplementary angles on a leisurely stroll.

2.1 PERPENDICULARITY

Objectives

After studying this chapter, you will be able to

- Recognize the need for clarity and concision in proofs
- Understand the concept of perpendicularity

Part One: Introduction

A Look Back and a Look Ahead

If you feel somewhat confused at this time, you need not feel discouraged. Some confusion is inevitable at the start of geometry. Be patient! Read the lessons carefully, study the sample problems closely, and the confusion will begin to go away. Also, see your teacher for help as you need it.

In Chapter 1, you concentrated on two-column proofs but were also exposed to paragraph proofs. When writing either type, remember that understanding what you are trying to say is the most important element.

From now on, when you write a two-column proof, try to state each reason in a single sentence or less. To help you, the problems in Problem Set A of this section and the next will include a hint when a proof requires more than two steps.

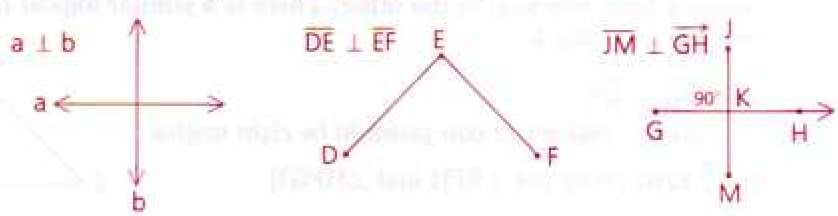
This chapter contains more definitions and theorems for you to memorize and use. Toward the end of the chapter, the proofs will begin to get a little longer. As the proofs become more challenging, you will find more satisfaction in completing them.

Perpendicular Lines, Rays, and Segments

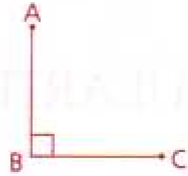
Perpendicularity, right angles, and 90° measurements all go together.

Definition Lines, rays, or segments that intersect at right angles are ***perpendicular*** (⊥).

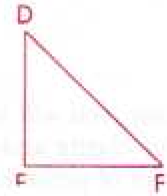
Below are some examples of perpendicularity.



In the figure at the right, the mark inside the angle (¬) indicates that ∠B is a right angle. It is also true that $\overline{AB} \perp \overline{BC}$ and ∠B = 90°.

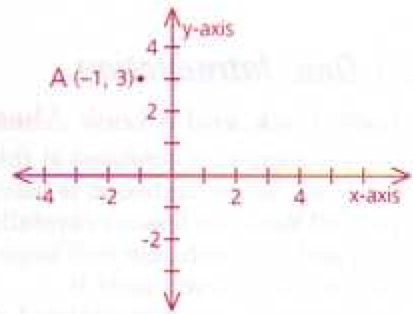


Do not assume perpendicularity from a diagram! In △DEF it *appears* that $\overline{DE} \perp \overline{EF}$, but we may not assume so.



In your algebra studies, you learned that two perpendicular number lines form a two-dimensional coordinate system, or coordinate plane. (The horizontal line is called the ***x-axis***; the vertical line, the ***y-axis***.) Each point on the plane can be represented by an ordered pair in the form (x, y). The values of x and y in the pair, called the point's ***coordinates***, represent the point's distances from the y-axis and the x-axis respectively. In the diagram, point A is represented by (−1, 3).

The intersection of the axes is called the ***origin***. Its coordinates are (0, 0).



Part Two: Sample Problems

Problem 1 *Given:* $\overline{AB} \perp \overline{BC}$, $\overline{DC} \perp \overline{BC}$

Conclusion: ∠B ≅ ∠C

Proof

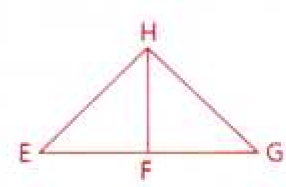
Statements	Reasons
1 $\overline{AB} \perp \overline{BC}$	1 Given
2 ∠B is a right angle.	2 If two segments are ⊥, they form a right angle.
3 $\overline{DC} \perp \overline{BC}$	3 Given
4 ∠C is a right angle.	4 Same as 2
5 ∠B ≅ ∠C	5 If angles are right angles, they are ≅.

The braces joining steps 1 and 2 emphasize the logical flow of reasoning from one step to the other. There is a similar logical flow from step 3 to step 4.

Problem 2 *Given:* $\overleftrightarrow{EH} \perp \overleftrightarrow{HG}$

Name all the angles you can prove to be right angles.

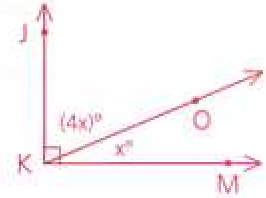
Answer Only ∠EHG (Why not ∠EFH and ∠HFG?)



Problem 3 *Given:* $\overrightarrow{KJ} \perp \overrightarrow{KM}$;

∠JKO is four times as large as ∠MKO.

Find: m∠JKO



Solution Since $\overrightarrow{KJ} \perp \overrightarrow{KM}$, m∠JKO + m∠MKO = 90.

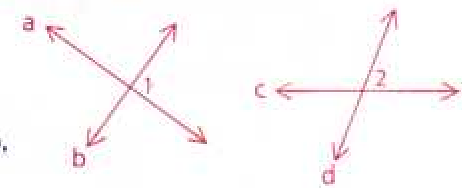
$$4x + x = 90$$
$$5x = 90$$
$$x = 18$$

Substituting 18 for x, we find that m∠JKO = 72.

Problem 4 *Given:* a ⊥ b,

c ⊥̸ d (c *is not* ⊥ *to* d.)

Conclusion: ∠1 ≅ ∠2

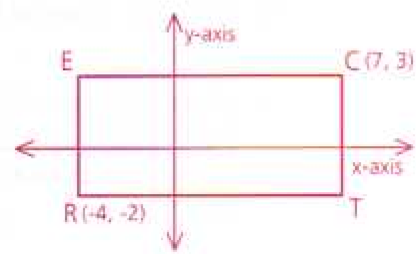


Solution This conclusion is false. Since a ⊥ b, ∠1 = 90°. Since c ⊥̸ d, ∠2 ≠ 90°. Since ∠1 and ∠2 have different measures, ∠1 ≇ ∠2.

Problem 5 *Given:* $\overline{EC}$ ∥ *to x-axis*

$\overline{RT}$ ∥ *to x-axis*

Find: Area of rectangle RECT

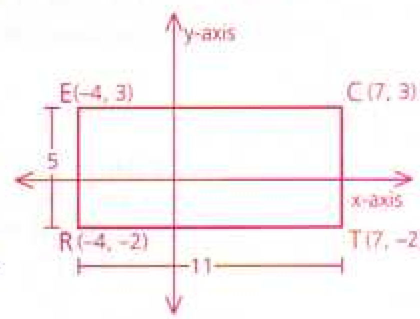


Solution The remaining coordinates are T = (7, −2) and E = (−4, 3). So RT = 11 and TC = 5 as shown. We shall concentrate on area in Chapter 12, but from previous courses you should know how to find a rectangle's area.

Area of rectangle = base × height

$$A = bh$$
$$= 11(5)$$
$$= 55$$

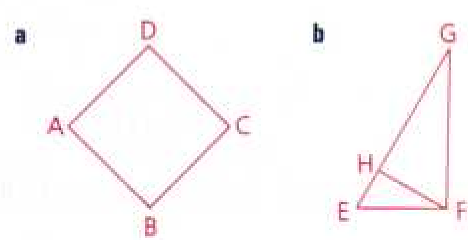
The area of RECT is 55 square units.



Part Three: Problem Sets

Problem Set A

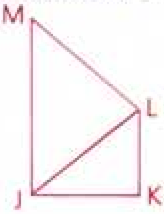
1 Name all the angles in the figures to the right that appear to be right angles.



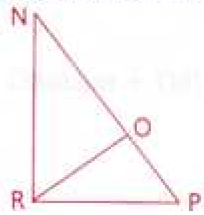
Problem Set A, *continued*

2 In each of the following, name the angles that can be *proved* to be right angles.

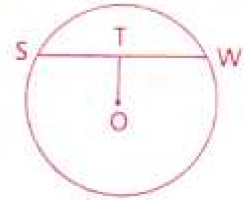
a Given: $\overline{JM} \perp \overline{JK}$



b Given: $\overrightarrow{RO} \perp \overrightarrow{PN}$

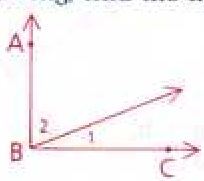


c Given: $\overline{OT} \perp \overline{SW}$

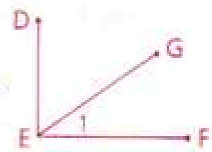


3 In each of the following, find the measure of $\angle 1$.

a $\overline{AB} \perp \overline{BC}$,
$\angle 2 = 68°17'34''$

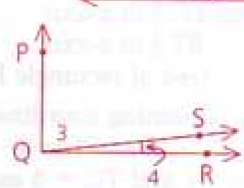


b $\overleftrightarrow{DE} \perp \overleftrightarrow{EF}$;
$\overrightarrow{EG}$ bisects $\angle DEF$.



4 **a** $\angle 1$ is five times as large as $\angle 2$. Find $m\angle 2$.

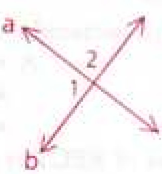
b $\angle 3$ is 72 times as large as $\angle 4$, and $\overleftrightarrow{PQ} \perp \overleftrightarrow{QR}$. Find $m\angle 4$ to the nearest tenth. (Hint: Use a calculator to do the arithmetic.)



5 On a graph, point A is at (0, 4). Point A is then rotated 90° clockwise about the origin to point A′. What are the coordinates of A′?

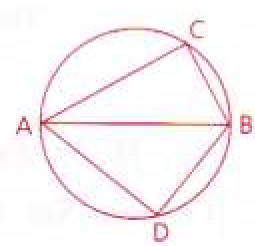
6 Given: $a \perp b$

Prove: $\angle 1 \cong \angle 2$ (Hint: This proof takes more than two steps. Remember, each reason should be a single sentence or less.)



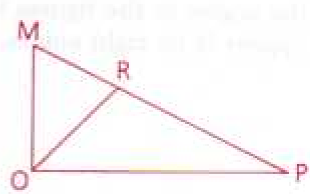
7 Given: $\angle ACB = 90°$,
$\overline{AD} \perp \overline{BD}$

Prove: $\angle C \cong \angle D$ (Hint: This proof takes more than three steps.)

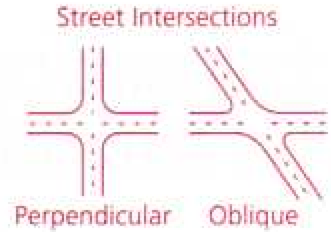


8 Given: $\angle MOR = (3x + 7)°$,
$\angle ROP = (4x - 1)°$,
$\overline{MO} \perp \overline{OP}$

Which angle is larger, $\angle MOR$ or $\angle ROP$?



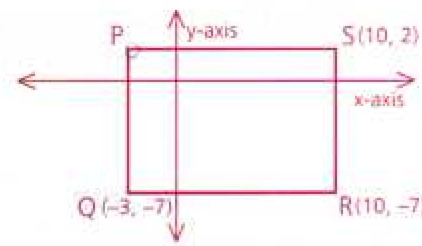
9 You are the engineer for the development of a new subdivision in your town. When you design your street intersections, is it better to make the intersections perpendicular or oblique? Explain why.
Note When two lines intersect and are not perpendicular, they are called ***oblique lines.***



10 PQRS is a rectangle.

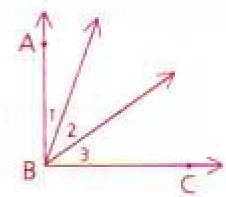
a Find the coordinates of point P.

b Find the area of the rectangle.



Problem Set B

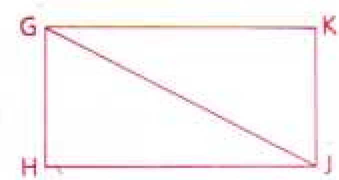
11 $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ and angles 1, 2, and 3 are in the ratio 1:2:3. Find the measure of each angle.



12 Line DE is perpendicular to line EF. The resulting angle is trisected, then one of the new angles is bisected, and then one of the resulting angles is trisected. How large is the smallest angle?

13 Given: $\angle HGJ = 37°20'$,
$\angle KGJ = 52°40'$,
$\overline{KJ} \perp \overline{HJ}$

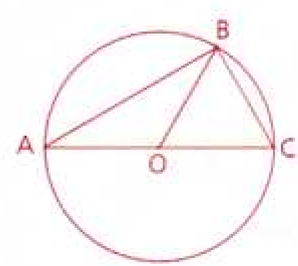
Conclusion: $\angle HGK \cong \angle HJK$ (Use a paragraph proof.)



Problem Set C

14 Given: $\overline{AB} \perp \overline{BC}$,
$\angle ABO = (2x + y)°$,
$\angle OBC = (6x + 8)°$,
$\angle AOB = (23y + 90)°$,
$\angle BOC = (4x + 4)°$

Find: $m\angle ABO$



15 If a ray, $\overrightarrow{BD}$, is chosen at random between the sides of $\angle ABC$, where $m\angle ABC = 100$, what is the probability that

a $\angle ABD$ is acute?

b $\angle DBC$ is acute?

c Both $\angle ABD$ and $\angle DBC$ are acute?

2.2 Complementary and Supplementary Angles

Objective

After studying this section, you will be able to

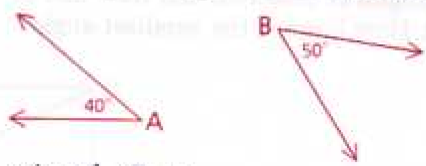
- Recognize complementary and supplementary angles

Part One: Introduction

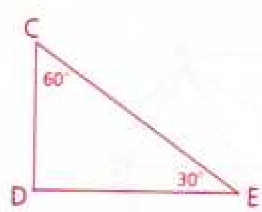
We frequently see pairs of angles whose measures add up to a right angle or a straight angle. In this section we will study such pairs of angles—those with sums of 90° and 180°.

Definition ***Complementary angles*** are two angles whose sum is 90° (a right angle). Each of the two angles is called the ***complement*** of the other.

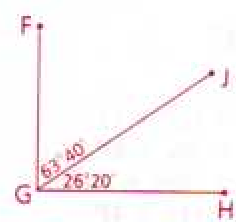
The following are examples of pairs of complementary angles.



∠A and ∠B are complementary.



∠C is comp. to ∠E.

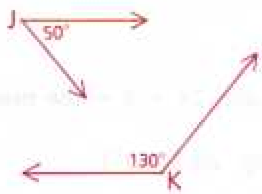


∠FGJ is the comp. of ∠JGH.

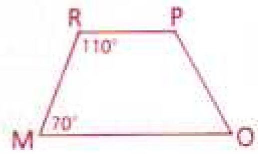
In the first diagram, ∠A is the complement of ∠B, and ∠B is the complement of ∠A. In the second diagram, two angles of a triangle, ∠C and ∠E, are complementary. In the third diagram, you can see how two complementary angles can share a side to form a right angle.

Definition ***Supplementary angles*** are two angles whose sum is 180° (a straight angle). Each of the two angles is called the ***supplement*** of the other.

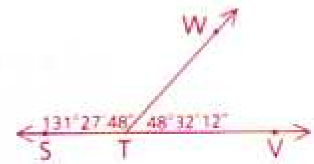
The following are examples of pairs of supplementary angles.



∠J and ∠K are supplementary.



∠M is supp. to ∠R.



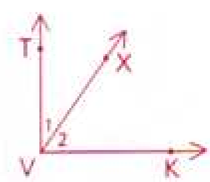
∠STW is the supp. of ∠WTV.

In the first diagram, ∠J is the supplement of ∠K, and vice versa. In the middle diagram, which angle is the supplement of ∠M?

Sometimes, two supplementary angles will form a straight angle by sharing a side. See if you can verify that ∠STW + ∠WTV = 180°.

Part Two: Sample Problems

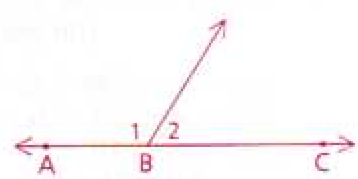
Problem 1 *Given: ∠TVK is a right ∠.*
Prove: ∠1 is comp. to ∠2.



Proof

Statements	Reasons
1 ∠TVK is a right ∠.	1 Given
2 ∠1 is comp. to ∠2.	2 If the sum of two ∠s is a right ∠, they are comp.

Problem 2 *Given: Diagram as shown*
Conclusion: ∠1 is supp. to ∠2.



Proof

Statements	Reasons
1 Diagram as shown	1 Given
2 ∠ABC is a straight angle.	2 Assumed from diagram
3 ∠1 is supp. to ∠2.	3 If the sum of two ∠s is a straight ∠, they are supp.

Problem 3 *The measure of one of two complementary angles is three greater than twice the measure of the other. Find the measure of each.*

Solution Draw the angles and place your algebra on the figure.

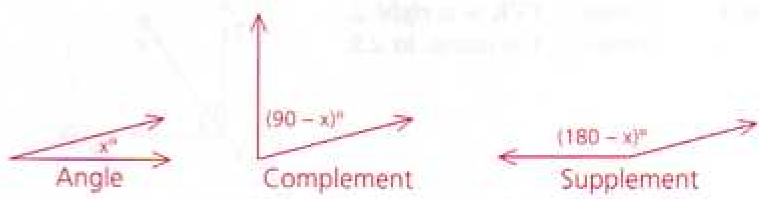
Let x = the measure of the smaller angle and 2x + 3 = the measure of the larger angle.

$$x + 2x + 3 = 90 \quad \text{(The sum of two comp. } \angle\text{s is } 90°.)$$
$$3x + 3 = 90$$
$$3x = 87$$
$$x = 29$$

The measure of one angle is 29. The measure of the other is 2(29) + 3, or 61.

Problem 4 *The measure of the supplement of an angle is 60 less than 3 times the measure of the complement of the angle. Find the measure of the complement.*

Solution Draw the three angles and place your algebra on the figure.



Let x = the measure of the angle.
So 90 − x = the measure of the complement.
(Do you know why?)
So 180 − x = the measure of the supplement.
(Do you know why?)

$$180 - x = 3(90 - x) - 60$$
$$180 - x = 270 - 3x - 60$$
$$180 - x = 210 - 3x$$
$$2x = 30$$
$$x = 15$$

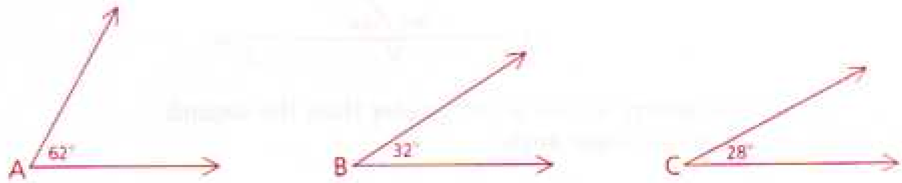
The measure of the complement is 90 − 15, or 75.

Note This is a key sample problem. The expressions used at the start of the solution (x, $90 - x$, and $180 - x$) are used in many problems throughout the book.

Part Three: Problem Sets

Problem Set A

1 Which two angles are complementary?



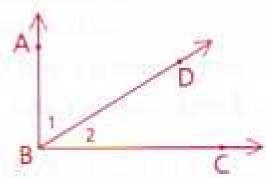
2 What is the supplement of a 70° angle?

3 ∠1 is complementary to ∠3. If ∠3 = $y°$, how large is ∠1?

4 Find the complement of a 61°21′13″ angle.

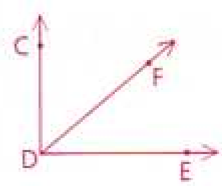
5 One of two complementary angles is twice the other. Find the measures of the angles.

6 Copy the figure and the proof below. Then complete the proof by filling in the missing statements.
Given: ∠1 is comp. to ∠2.
Prove: $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$

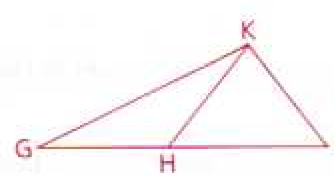


Statements	Reasons
1 ________	1 Given
2 ________	2 If a ray divides an ∠ into two comp. ∠s, then the original ∠ is a right ∠.
3 ________	3 If two lines intersect to form a right ∠, the two lines are ⊥.

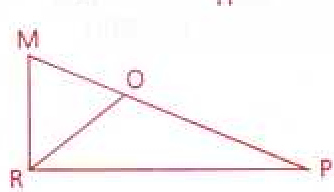
7 Given: $\overleftrightarrow{CD} \perp \overleftrightarrow{DE}$
Prove: ∠CDF is comp. to ∠FDE. (Hint: This proof takes more than two steps.)



8 Given: Diagram as shown
Prove: ∠GHK is supp. to ∠KHJ. (Hint: This proof takes more than two steps.)

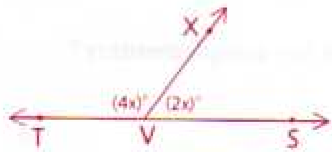


9 Given: ∠MRO is comp. to ∠PRO.
Prove: ∠MRP is a right angle.



Problem Set A, *continued*

10 Find the measure of ∠XVS.



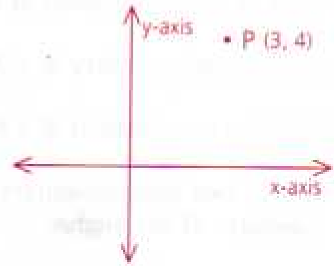
11 One of two supplementary angles is 70° greater than the second. Find the measure of the larger angle.

Problem Set B

12 **a** Point P is reflected over the y-axis to point A. Find the coordinates of A.

b Point P is reflected over the origin to point B. Find the coordinates of B.

c If C is the midpoint of $\overline{PA}$, find the coordinates of C.

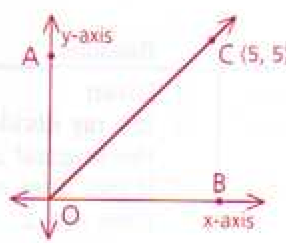


13 Complete each of the following conditional statements and justify your completion with an explanation.

a If two angles are supplementary and congruent, then __?__.

b If two angles are complementary and congruent, then __?__.

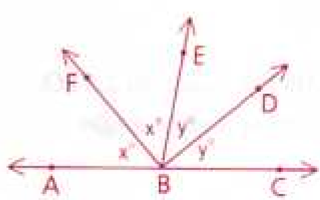
14 Find the measures of ∠AOC and ∠COB in the graph.



15 Find, to the nearest hundredth, the area of the rectangle.

16 Two supplementary angles are in the ratio 11:7. Find the measure of each.

17 Write a paragraph proof to show that ∠ABF is complementary to ∠EBD.

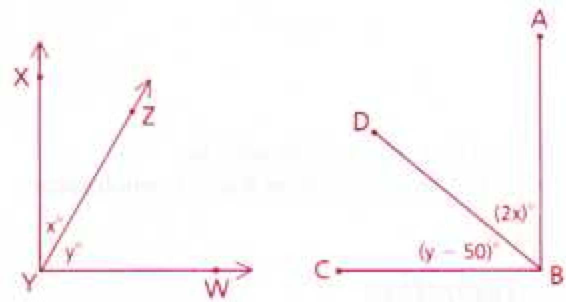


18 The larger of two supplementary angles exceeds 7 times the smaller by 4°. Find the measure of the larger angle.

19 One of two complementary angles added to one-half the other yields 72°. Find half the measure of the larger.

20 Given: $\overline{XY} \perp \overline{YW}$, $\overline{AB} \perp \overline{BC}$

Find: m∠DBC



21 The supplement of an angle is four times the complement of the angle. Find the measure of the complement.

22 Five times the complement of an angle less twice the angle's supplement is 40°. Find the measure of the supplement.

23 The measure of the supplement of an angle is 30° less than five times the measure of the complement. Find two-fifths the measure of the complement.

24 Arnex has a 30°, a 60°, a 150°, a 45°, and a 135° angle in his pocket. He takes out two of the five angles. Find the probability that

a The two angles are supplementary

b The two angles are complementary

Problem Set C

25 The supplement of an angle is 60° less than twice the supplement of the complement of the angle. Find the measure of the complement.

26 Debbie has drawn distinct rays $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, $\overrightarrow{BE}$, and $\overrightarrow{BF}$ on a piece of paper, with ∠ABC being a straight angle.

a What is the minimum number of pairs of complementary angles that she could have drawn?

b What is the maximum number of pairs of complementary angles that she could have drawn?

c What is the minimum number of pairs of supplementary angles that she could have drawn?

d What is the maximum number of pairs of supplementary angles that she could have drawn?

2.3 Drawing Conclusions

Objective

After studying this section, you will be able to

- Follow a five-step procedure to draw logical conclusions

Part One: Introduction

There wouldn't be much progress in this world if all we did was justify conclusions that someone else had already drawn. Neither will you make much progress as a student of geometry if all you can do is justify conclusions the textbook has already stated. Although the following procedure may not work every time, it will be helpful to you in drawing conclusions.

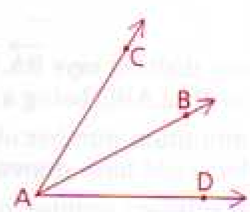
Procedure for Drawing Conclusions

1. Memorize theorems, definitions, and postulates.
2. Look for key words and symbols in the given information.
3. Think of all the theorems, definitions, and postulates that involve those keys.
4. Decide which theorem, definition, or postulate allows you to draw a conclusion.
5. Draw a conclusion, and give a reason to justify the conclusion. Be certain that you have not used the *reverse* of the correct reason.

Example

Given: $\overrightarrow{AB}$ *bisects* $\angle CAD$.

Conclusion: ___?___



Thinking Process:

The key word is *bisects*.

The key symbols are $\rightarrow$ and $\angle$.

The definition of *bisector* (of an angle) contains those keys.

An appropriate conclusion is that $\angle CAB \cong \angle DAB$.

Statements	Reasons
1 $\overrightarrow{AB}$ bisects ∠CAD.	1 Given
2 ∠CAB ≅ ∠DAB	2 If a ray bisects an ∠, then it divides the ∠ into two ≅ angles.

Note The "If . . ." part of the reason matches the given information, and the "then . . ." part matches the conclusion being justified. *Be sure not to reverse that order.*

Part Two: Sample Problems

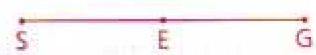
For each of these problems, we will write a two-column proof, supplying a correct conclusion and reason.

Problem 1 *Given: ∠A is a right angle.*
∠B is a right angle.
Conclusion: ___?___

Proof

Statements	Reasons
1 ∠A is a right angle.	1 Given
2 ∠B is a right angle.	2 Given
3 ∠A ≅ ∠B	3 If two ∠s are right ∠s, then they are ≅.

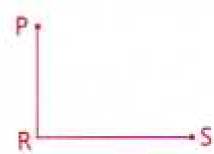
Problem 2 *Given: E is the midpoint of* $\overline{SG}$.
Conclusion: ___?___



Proof

Statements	Reasons
1 E is the midpoint of $\overline{SG}$.	1 Given
2 $\overline{SE} \cong \overline{EG}$	2 If a point is the midpoint of a segment, the point divides the segment into two ≅ segments.

Problem 3 *Given: ∠PRS is a right angle.*
Conclusion: ___?___



Proof

Statements	Reasons
1 ∠PRS is a right ∠.	1 Given
2 $\overleftrightarrow{PR} \perp \overleftrightarrow{RS}$	2 If two lines intersect to form a right ∠, they are ⊥ .

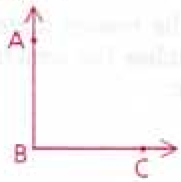
In sample problem 3, we could have drawn a different conclusion. Do you know what that other conclusion is?

Part Three: Problem Sets

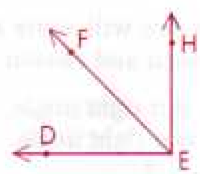
Problem Set A

In problems 1–7, write a two-column proof, supplying your own correct conclusion and reason.

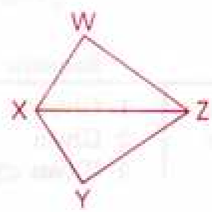
1 Given: $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$
Conclusion: ___?___



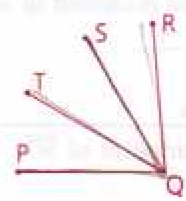
2 Given: ∠DEF is comp. to ∠HEF.
Conclusion: ___?___



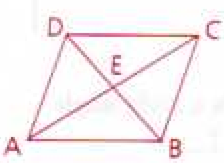
3 Given: ∠WXZ ≅ ∠YXZ
Conclusion: ___?___



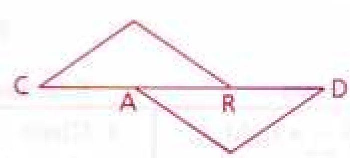
4 Given: $\overrightarrow{QS}$ and $\overrightarrow{QT}$ trisect ∠PQR.
Conclusion: ___?___



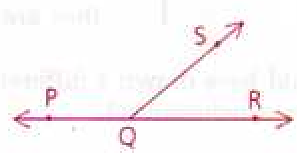
5 Given: E is the midpoint of $\overline{AC}$.
Conclusion: ___?___



6 Given: A and R trisect $\overline{CD}$.
Conclusion: ___?___



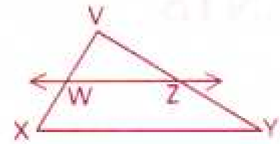
7 Given: Diagram as shown
Conclusion: ___?___



Problem Set B

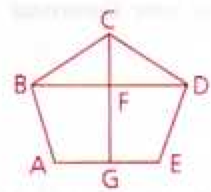
In problems 8–12, draw at least two conclusions for each "given" statement, and give reasons to support them in two-column-proof form.

8 Given: $\overleftrightarrow{WZ}$ bisects $\overline{VY}$.
Conclusions: ?

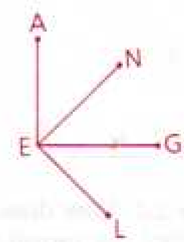


9 Given: $\overline{PA} \perp \overline{AR}$
Conclusions: ?

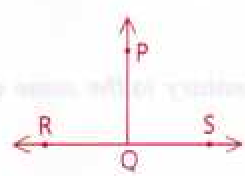
10 Given: $\overleftrightarrow{CG}$ bisects $\overline{BD}$.
Conclusions: ?



11 Given: $\angle AEN \cong \angle GEN \cong \angle GEL$
Conclusions: ?

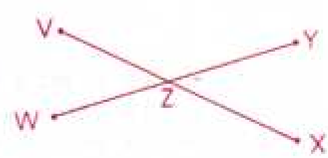


12 Given: $m\angle PQS = 90$
Conclusions: ?



Problem Set C

13 Given: Two intersecting lines as shown
Conclusions: ? (Find as many as you can.)



14 The right angle of a right triangle is bisected. Draw a diagram and set up the given information. Then discuss all possible conclusions.

2.4

Congruent Supplements and Complements

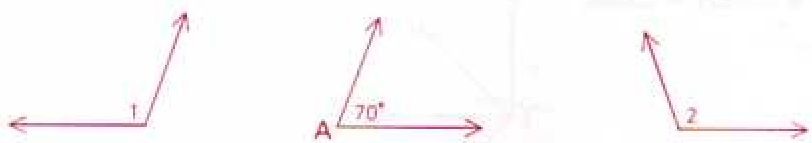
Objective

After studying this section, you will be able to

- Prove angles congruent by means of four new theorems

Part One: Introduction

In the diagram below, ∠1 is supplementary to ∠A, and ∠2 is also supplementary to ∠A.

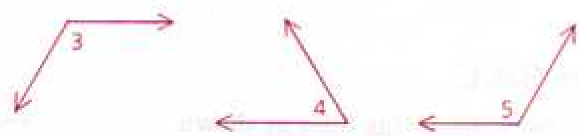


How large is ∠1? Now calculate ∠2. How does ∠1 compare with ∠2? Your results will illustrate (but not prove) the following theorem.

Theorem 4 ***If angles are supplementary to the same angle, then they are congruent.***

Given: ∠3 is supp. to ∠4.
∠5 is supp. to ∠4.

Prove: ∠3 ≅ ∠5



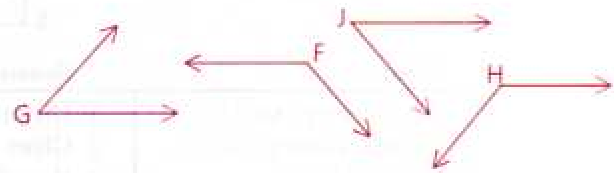
Proof: ∠3 is supp. to ∠4, so m∠3 + m∠4 = 180.
Therefore, m∠3 = 180 − m∠4.
∠5 is supp. to ∠4, so m∠5 + m∠4 = 180.
Therefore, m∠5 = 180 − m∠4.
Since ∠3 and ∠5 have the same measure, ∠3 ≅ ∠5.

A companion to Theorem 4 follows.

Theorem 5 ***If angles are supplementary to congruent angles, then they are congruent.***

Given: ∠F is supp. to ∠G.
∠H is supp. to ∠J.
∠G ≅ ∠J

Conclusion: ∠F ≅ ∠H



The proof of Theorem 5 is similar to that of Theorem 4.

Two similar theorems apply to complementary angles.

Theorem 6 ***If angles are complementary to the same angle, then they are congruent.***

Theorem 7 ***If angles are complementary to congruent angles, then they are congruent.***

When studying the definitions of such terms as *right angle*, *bisect*, *midpoint*, and *perpendicular*, you will master the concepts more quickly if you try to understand the ideas involved without memorizing the definitions word for word. The theorems in this section, however, are different. Unless you *memorize* Theorems 4–7, you will have difficulty remembering the concepts they contain.

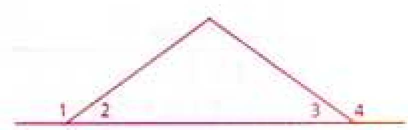
Therefore, before you begin your homework,

1 Memorize Theorems 4–7
2 Read the sample problems carefully, so that you understand which of the theorems is used in each type of problem

Part Two: Sample Problems

Problem 1 *Given:* ∠1 is supp. to ∠2.
∠3 is supp. to ∠4.
∠1 ≅ ∠4

Conclusion: ∠2 ≅ ∠3

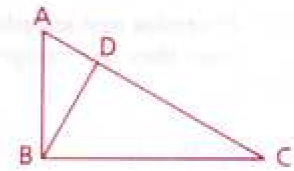


Proof

Statements	Reasons
1 ∠1 is supp. to ∠2.	1 Given
2 ∠3 is supp. to ∠4.	2 Given
3 ∠1 ≅ ∠4	3 Given
4 ∠2 ≅ ∠3	4 If angles are supplementary to ≅ angles, they are ≅. (Short form: Supplements of ≅ ∠s are ≅.)

Problem 2 *Given:* ∠A *is comp. to* ∠C.
∠DBC *is comp. to* ∠C.

Conclusion: ___?___

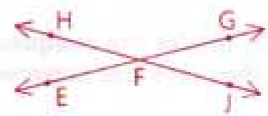


Proof

Statements	Reasons
1 ∠A is comp. to ∠C.	1 Given
2 ∠DBC is comp. to ∠C.	2 Given
3 ∠A ≅ ∠DBC	3 If angles are complementary to the same angle, they are ≅. (Short form: Complements of the same ∠ are ≅.)

Problem 3 *Given: Diagram as shown*

Prove: ∠HFE ≅ ∠GFJ

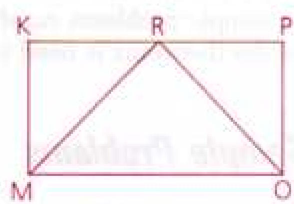


Proof

Statements	Reasons
1 Diagram as shown.	1 Given
2 ∠EFG is a straight ∠.	2 Assumed from diagram
3 ∠HFE is supp. to ∠HFG.	3 If two ∠s form a straight ∠, they are supplementary.
4 ∠HFJ is a straight ∠.	4 Same as 2
5 ∠GFJ is supp. to ∠HFG.	5 Same as 3
6 ∠HFE ≅ ∠GFJ	6 If angles are supplementary to the same angle, they are ≅. (Short form: Supplements of the same ∠ are ≅.)

Problem 4 *Given:* $\overline{KM} \perp \overline{MO}$,
$\overline{PO} \perp \overline{MO}$,
∠KMR ≅ ∠POR

Prove: ∠ROM ≅ ∠RMO



Proof

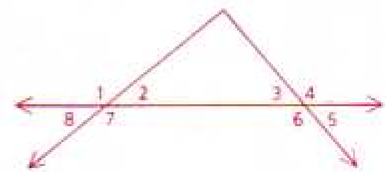
Statements	Reasons
1 $\overline{KM} \perp \overline{MO}$	1 Given
2 ∠KMO is a right ∠.	2 If segments are ⊥, they form right ∠s.
3 ∠RMO is comp. to ∠KMR.	3 If two ∠s form a right ∠, they are complementary.
4 In a similar manner, ∠ROM is comp. to ∠POR.	4 Reasons 1–3
5 ∠KMR ≅ ∠POR	5 Given
6 ∠ROM ≅ ∠RMO	6 If angles are complementary to ≅ angles, they are ≅. (Short form: Complements of ≅ ∠s are ≅.)

Part Three: Problem Sets

Problem Set A

Before starting the assignment, memorize Theorems 4–7. The key to the use of these theorems is to look for the double use of the word *complementary* or *supplementary* in a problem.

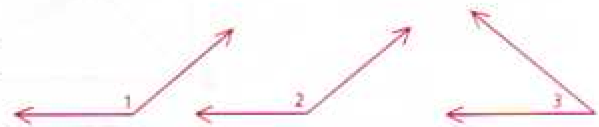
1 Given: ∠2 is comp. to ∠3.
∠4 = 131°



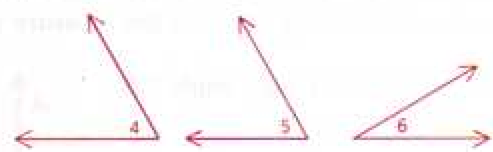
Find the measure of each of the following angles.

a ∠3 **b** ∠6 **c** ∠5 **d** ∠2 **e** ∠1 **f** ∠8 **g** ∠7

2 Given: ∠1 is supp. to ∠3.
∠2 is supp. to ∠3.
Prove: ∠1 ≅ ∠2



3 Given: ∠4 is comp. to ∠6.
∠5 is comp. to ∠6.
Prove: ∠4 ≅ ∠5

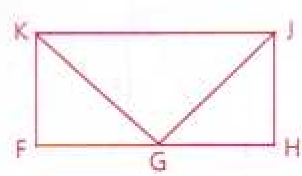


4 One of two supplementary angles is four times the other. Find the larger angle.

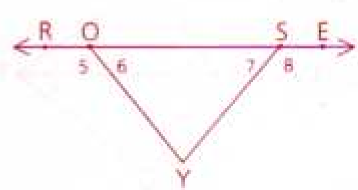
5 One of two complementary angles is 20° larger than the other. Find the measure of each.

6 Given: ∠4 is supp. to ∠6.
∠5 is supp. to ∠7.
∠4 ≅ ∠5
Conclusion: __?__

7 Given: ∠FKJ is a right ∠.
∠HJK is a right ∠.
∠GKJ ≅ ∠GJK
Conclusion: ∠FKG ≅ ∠HJG

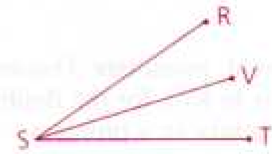


8 Given: Diagram as shown,
∠6 ≅ ∠7
Prove: ∠5 ≅ ∠8



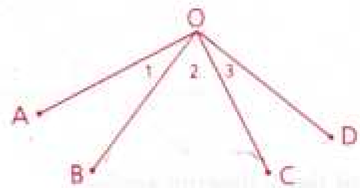
Problem Set A, *continued*

9 Given: $\overrightarrow{SV}$ bisects ∠RST.
Conclusion: ∠RSV ≅ ∠TSV

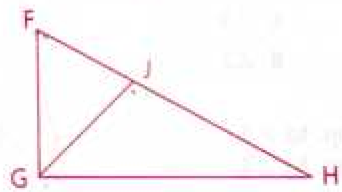


Problem Set B

10 Given: $\overleftrightarrow{OA} \perp \overleftrightarrow{OC}$,
$\overleftrightarrow{OB} \perp \overleftrightarrow{OD}$
Prove: ∠1 ≅ ∠3

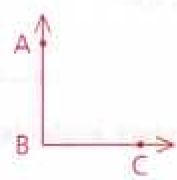


11 Given: ∠F is comp. to ∠FGJ.
∠H is comp. to ∠HGJ.
$\overrightarrow{GJ}$ bisects ∠FGH.
Conclusion: ∠F ≅ ∠H



12 The measure of the supp. of an ∠ exceeds 3 times the measure of the comp. of the ∠ by 10. Find the measure of the comp.

13 Draw the reflection of right angle ABC over line $\overleftrightarrow{AB}$.

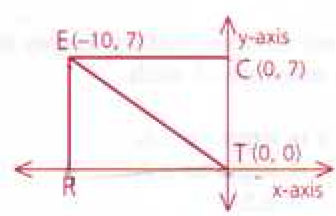


14 RECT is a rectangle.

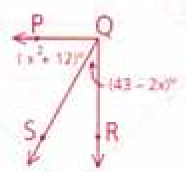
a Find the coordinates of R.

b What do we know about ∠RTE and ∠CTE?

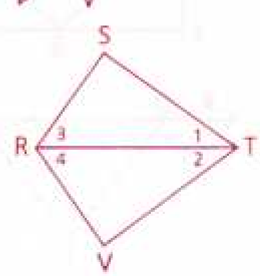
c Find the area of △ERT.



15 Given: $\overline{PQ} \perp \overline{QR}$
Find: m∠PQS



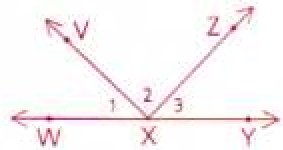
16 Given: ∠1 is comp. to ∠4.
∠2 is comp. to ∠3.
$\overrightarrow{RT}$ bisects ∠SRV.
Prove: $\overrightarrow{TR}$ bisects ∠STV.



17 If three times the supp. of an ∠ is subtracted from seven times the comp. of the ∠, the answer is the same as that obtained by trisecting a right ∠. Find the supplement.

18 Given: ∠WXZ ≅ ∠VXY

Conclusion: ∠1 ≅ ∠3



19 Given: ∠PQR supp. ∠QRS, ∠QRS supp. ∠TWX,

∠PQR = $(5x - 48)°$, ∠TWX = $(2x + 30)°$

Find: m∠QRS

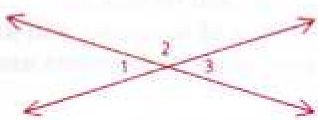
Problem Set C

20 Given: ∠1 = $(x^2 + 3y)°$,

∠2 = $(20y + 3)°$,

∠3 = $(3y + 4x)°$

Find: m∠1



21 The ratio of an angle to its supplement is 3:7. Find the ratio of the angle to its complement.

MATHEMATICAL EXCURSION

GEOMETRY IN COMPUTERS

Three-Dimensional views on a flat screen

Designers, architects, and draftspeople are putting away their T squares and doing more of their work with computers. A wide variety of software for computer-aided drafting and design (CADD) has made it possible to do accurate work on a computer screen. Using a computer makes exploring solutions to design problems, as well as making corrections and revising, more efficient. A computer also performs calculations and offers a system for filing alternative versions of a plan.

One of the most exciting features of CADD software is that it allows you to create a three-dimensional design and then rotate it on the screen, still in three dimensions. This enables an architect or designer to see, with the press of a key or the click of a mouse, how his or her design would look from any direction or angle.

Using a CADD program, you can see the measure of an angle displayed as you draw the angle. You can instruct the program to automatically bisect an angle you have drawn.

Simpler geometric drawing programs such as The Geometric Supposer offer some of the drawing and measuring capabilities of the CADD programs, including the opportunity for experimenting with geometric concepts such as angle sizes and relationships.

2.5 ADDITION AND SUBTRACTION PROPERTIES

Objectives

After studying this section, you will be able to

- Apply the addition properties of segments and angles
- Apply the subtraction properties of segments and angles

Part One: Introduction

Addition Properties

In the diagram below, AB = CD. Do you think that AC = BD? Suppose that BC were 3 cm. Would AC = BD? If AB = CD, does the length of BC have any effect on whether AC = BD?

Your answers should be that AC = BD in each case and the length of BC does not effect that equality. This is a geometric application of the algebraic Addition Property of Equality (AB + BC = CD + BC).

Theorem 8 ***If a segment is added to two congruent segments, the sums are congruent. (Addition Property)***

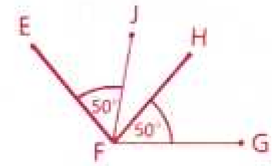
Given: $\overline{PQ} \cong \overline{RS}$

Conclusion: $\overline{PR} \cong \overline{QS}$

Proof: $\overline{PQ} \cong \overline{RS}$, so by definition of congruent segments, PQ = RS. Now, the Addition Property of Equality says that we may add QR to both sides, so PQ + QR = RS + QR. Substituting, we get PR = QS. Therefore, $\overline{PR} \cong \overline{QS}$ by the definition of congruent segments (reversed).

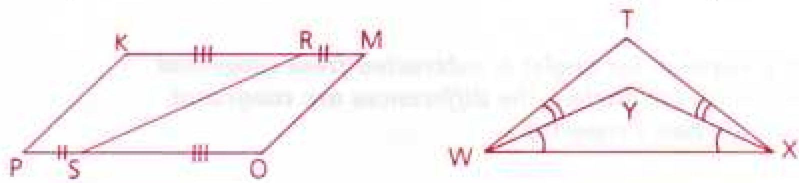
Does a similar relationship hold for angles? Is ∠EFH necessarily congruent to ∠JFG?

The next theorem confirms that the answer is yes. Its proof is like that of Theorem 8.



Theorem 9 ***If an angle is added to two congruent angles, the sums are congruent. (Addition Property)***

In the figures below, identical tick marks indicate congruent parts.



Do you think that $\overline{KM}$ is necessarily congruent to $\overline{PO}$? In the right-hand diagram, is ∠TWX necessarily congruent to ∠TXW? The answer to these questions is yes.

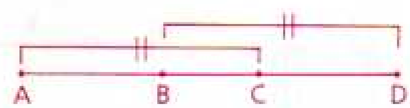
These congruencies are established by the following two theorems. Their proofs are similar to that of Theorem 8.

Theorem 10 ***If congruent segments are added to congruent segments, the sums are congruent. (Addition Property)***

Theorem 11 ***If congruent angles are added to congruent angles, the sums are congruent. (Addition Property)***

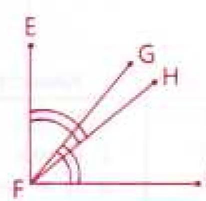
Subtraction Properties

We now have four addition properties. Because subtraction is equivalent to addition of an opposite, we can expect four corresponding subtraction properties.



If AC = BD, is AB = CD?

Let AC = 12 and BC = 3. How long is $\overline{BD}$? Is AB = CD?

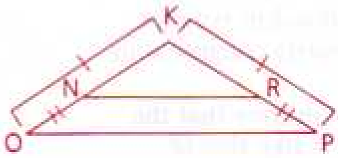


If ∠EFH ≅ ∠GFJ, is ∠EFG ≅ ∠HFJ?

Let m∠EFH = 50 and m∠GFH = 10. How large is ∠GFJ? Is ∠EFG ≅ ∠HFJ?

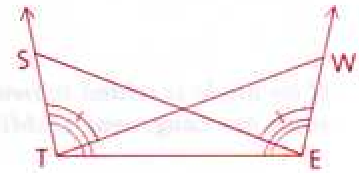
If KO = KP and NO = RP, is KN = KR?

Try this on your own and see what you think.



If ∠STE ≅ ∠WET and ∠STW ≅ ∠WES, is ∠WTE ≅ ∠SET?

Try this on your own.



Your results should agree with the next two theorems.

Theorem 12 ***If a segment (or angle) is subtracted from congruent segments (or angles), the differences are congruent. (Subtraction Property)***

Theorem 13 ***If congruent segments (or angles) are subtracted from congruent segments (or angles), the differences are congruent. (Subtraction Property)***

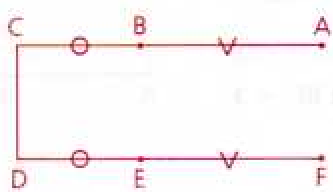
Using the Addition and Subtraction Properties in Proofs

1 An addition property is used when the segments or angles in the conclusion are *greater than* those in the given information.
2 A subtraction property is used when the segments or angles in the conclusion are *smaller than* those in the given information.

Part Two: Sample Problems

Problem 1 Given: $\overline{AB} \cong \overline{FE}$, $\overline{BC} \cong \overline{ED}$

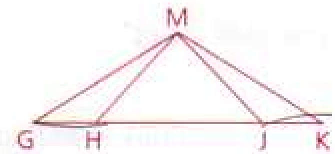
Prove: $\overline{AC} \cong \overline{FD}$



Proof

Statements	Reasons
1 $\overline{AB} \cong \overline{FE}$	1 Given
2 $\overline{BC} \cong \overline{ED}$	2 Given
3 $\overline{AC} \cong \overline{FD}$	3 If ≅ segments are added to ≅ segments, the sums are ≅. (Addition Property)

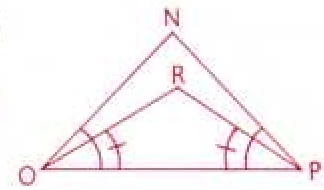
Problem 2 *Given:* $\overline{GJ} \cong \overline{HK}$
Conclusion: $\overline{GH} \cong \overline{JK}$



Proof

Statements	Reasons
1 $\overline{GJ} \cong \overline{HK}$	1 Given
2 $\overline{GH} \cong \overline{JK}$	2 If a segment ($\overline{HJ}$) is subtracted from ≅ segments, the differences are ≅. (Subtraction Property)

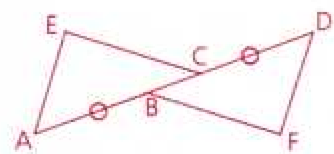
Problem 3 *Given:* ∠NOP ≅ ∠NPO,
∠ROP ≅ ∠RPO
Prove: ∠NOR ≅ ∠NPR



Proof

Statements	Reasons
1 ∠NOP ≅ ∠NPO	1 Given
2 ∠ROP ≅ ∠RPO	2 Given
3 ∠NOR ≅ ∠NPR	3 If ≅ angles are subtracted from ≅ angles, the differences are ≅. (Subtraction Property)

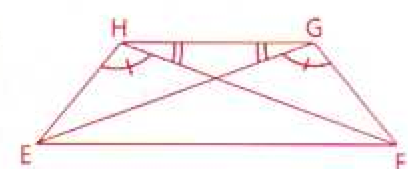
Problem 4 *Given:* $\overline{AB} \cong \overline{CD}$
Conclusion: __?__



Proof

Statements	Reasons
1 $\overline{AB} \cong \overline{CD}$	1 Given
2 $\overline{AC} \cong \overline{BD}$	2 If a segment ($\overline{BC}$) is added to ≅ segments, the sums are ≅. (Addition Property)

Problem 5 *Given:* ∠HEF *is supp. to* ∠EHG.
∠GFE *is supp. to* ∠FGH.
∠EHF ≅ ∠FGE,
∠GHF ≅ ∠HGE
Conclusion: ∠HEF ≅ ∠GFE



Proof

Statements	Reasons
1 ∠HEF is supp. to ∠EHG.	1 Given
2 ∠GFE is supp. to ∠FGH.	2 Given
3 ∠EHF ≅ ∠FGE	3 Given
4 ∠GHF ≅ ∠HGE	4 Given
5 ∠EHG ≅ ∠FGH	5 If ≅ angles are added to ≅ angles, the sums are ≅. (Addition Property)
6 ∠HEF ≅ ∠GFE	6 Supplements of ≅ ∠s are ≅.

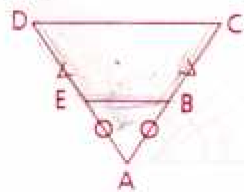
Part Three: Problem Sets

Problem Set A

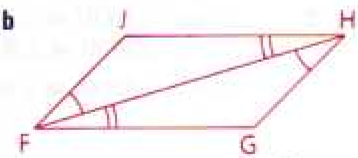
Throughout this problem set, think of addition when you are asked to prove that segments or angles are larger than the given segments or angles. Think of subtraction when you are asked to prove that segments or angles are smaller than the given segments or angles.

1 Name the angles or segments that are congruent by the Addition Property.

a

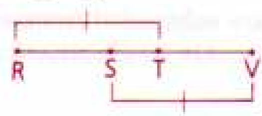


b

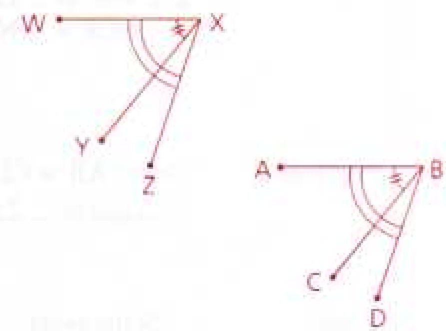


2 Name the angles or segments that are congruent by the Subtraction Property.

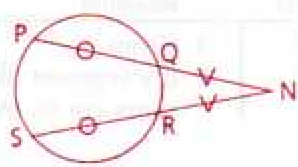
a



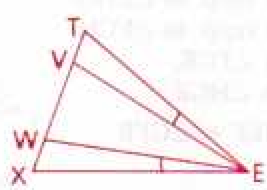
b



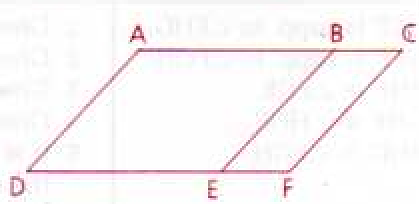
3 Given: $\overline{PQ} \cong \overline{SR}$,
$\overline{QN} \cong \overline{RN}$
Conclusion: $\overline{PN} \cong \overline{SN}$



4 Given: $\angle TEV \cong \angle XEW$
Prove: $\angle TEW \cong \angle XEV$



5 Given: $\overline{AC} \cong \overline{DF}$,
$\overline{BC} \cong \overline{EF}$
Prove: $\overline{AB} \cong \overline{DE}$

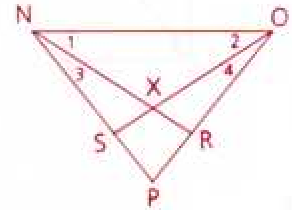


6 Given: $\overline{GH} \cong \overline{JK}$, GH = x + 10,
HJ = 8, JK = 2x − 4

Find: GJ

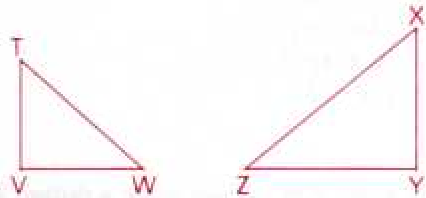
7 Given: ∠PNO ≅ ∠PON,
∠1 ≅ ∠2

Conclusion: __?__



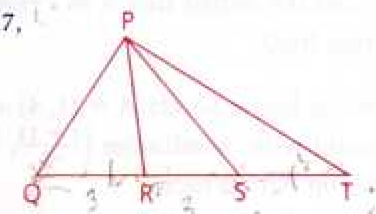
8 Given: ∠T is comp. to ∠W.
∠X is comp. to ∠Z.
∠Z ≅ ∠W

Prove: __?__



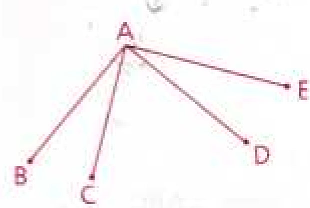
9 Given: $\overline{QR} \cong \overline{ST}$, QS = 5x + 17,
RT = 10 − 2x, RS = 3

Find: QS and QT



10 Given: ∠BAD is a right ∠.
$\overline{CA} \perp \overline{AE}$

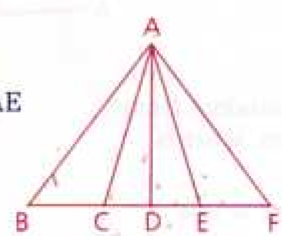
Prove: ∠BAC ≅ ∠EAD



Problem Set B

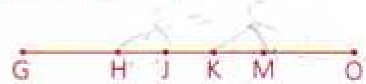
11 Given: ∠BAD ≅ ∠FAD;
$\overrightarrow{AD}$ bisects ∠CAE.

Conclusion: ∠BAC ≅ ∠FAE



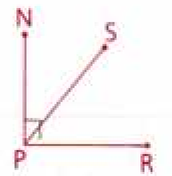
12 Given: J and K are trisection
points of $\overline{HM}$.
$\overline{GH} \cong \overline{MO}$

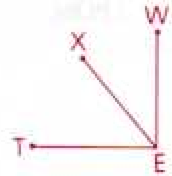
Conclusion: $\overline{GJ} \cong \overline{KO}$



Problem Set B, *continued*

13 Given: ∠NPR is a right ∠.
$\overline{WE} \perp \overline{ET}$,
∠SPR ≅ ∠XET
Prove: ∠NPS ≅ ∠WEX





14 Given: ∠A is comp. to ∠B.
∠C is comp. to ∠B.
∠A = $(3x + y)°$,
∠B = $(x + 4y + 2)°$,
∠C = $(3y - 3)°$
Find: m∠B

15 Draw a right angle ABC. Then draw a dotted line such that the reflection of $\overrightarrow{BA}$ over the dotted line is $\overrightarrow{BC}$. How would you describe this dotted line?

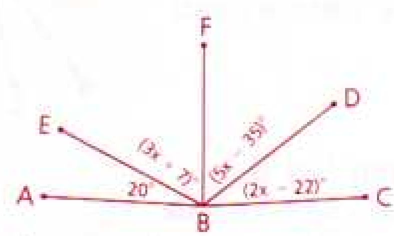
16 On a graph, carefully locate points A = (1, 4) and B = (11, 10). Now locate the point with coordinates $\left(\frac{1 + 11}{2}, \frac{10 + 4}{2}\right)$. Does this point appear to be on $\overline{AB}$? Where?

Problem Set C

17 $\overrightarrow{BF}$ bisects ∠DBE.

a Does $\overrightarrow{BF}$ bisect ∠CBA?

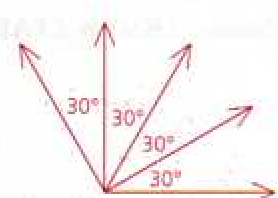
b What did you discover about ∠ABC and $\overrightarrow{BF}$?



18 If two angles are chosen at random from the ten angles in the diagram, what is the probability that

a The sum of their measures is less than 90?

b They are complementary?



19 Find the measure of the angle formed by the hands of a clock at 5:55 A.M.

2.6 Multiplication and Division Properties

Objective

After studying this section, you will be able to

- Apply the multiplication and division properties of segments and angles

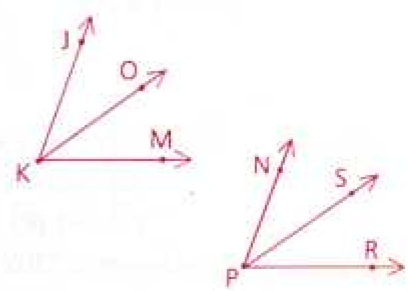
Part One: Introduction

In the figure below, B, C, F, and G are trisection points.

If AB = EF = 3, what can we say about $\overline{AD}$ and $\overline{EH}$?

If $\overline{AB} \cong \overline{EF}$, is $\overline{AD}$ congruent to $\overline{EH}$?

In the figure at the right, $\overrightarrow{KO}$ and $\overrightarrow{PS}$ are angle bisectors.



If m∠JKO = m∠NPS = 25, what can we say about ∠JKM and ∠NPR?

If ∠JKO ≅ ∠NPS, is ∠JKM congruent to ∠NPR?

The examples above illustrate a property whose proof is similar to the proof of Theorem 8.

Theorem 14 ***If segments (or angles) are congruent, their like multiples are congruent. (Multiplication Property)***

Also, because division is equivalent to multiplication by the reciprocal of the divisor, it is easy to prove the next theorem.

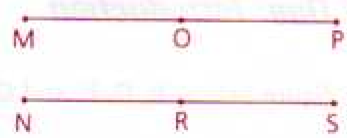
Theorem 15 ***If segments (or angles) are congruent, their like divisions are congruent. (Division Property)***

Using the Multiplication and Division Properties in Proofs

1. Look for a double use of the word *midpoint* or *trisect* or *bisects* in the given information.
2. The Multiplication Property is used when the segments or angles in the conclusion are *greater than* those in the given information.
3. The Division Property is used when the segments or angles in the conclusion are *smaller than* those in the given information.

Part Two: Sample Problems

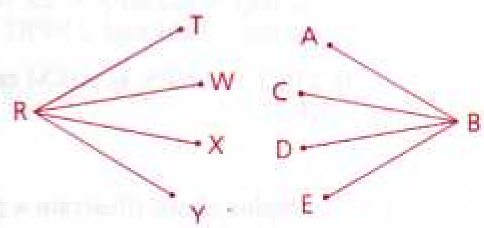
Problem 1 *Given:* $\overline{MP} \cong \overline{NS}$;
O *is the midpoint of* $\overline{MP}$.
R *is the midpoint of* $\overline{NS}$.
Prove: $\overline{MO} \cong \overline{NR}$



Proof

Statements	Reasons
1 $\overline{MP} \cong \overline{NS}$	1 Given
2 O is the midpoint of $\overline{MP}$.	2 Given
3 R is the midpoint of $\overline{NS}$.	3 Given
4 $\overline{MO} \cong \overline{NR}$	4 If segments are ≅, their like divisions (halves) are ≅. (Division Property)

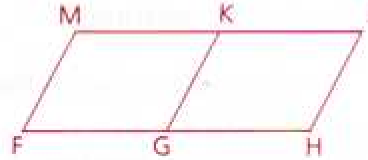
Problem 2 *Given:* $\angle TRY \cong \angle ABE$;
$\overrightarrow{RW}$ *and* $\overrightarrow{RX}$ *trisect* $\angle TRY$.
$\overrightarrow{BC}$ *and* $\overrightarrow{BD}$ *trisect* $\angle ABE$.
Conclusion: $\angle TRW \cong \angle CBD$



Proof

Statements	Reasons
1 $\angle TRY \cong \angle ABE$	1 Given
2 $\overrightarrow{RW}$ and $\overrightarrow{RX}$ trisect $\angle TRY$.	2 Given
3 $\overrightarrow{BC}$ and $\overrightarrow{BD}$ trisect $\angle ABE$.	3 Given
4 $\angle TRW \cong \angle CBD$	4 If angles are ≅, their like divisions (thirds) are ≅. (Division Property)

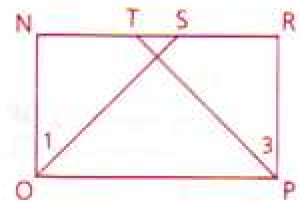
Problem 3 Given: $\overline{MK} \cong \overline{FG}$;
$\overline{KG}$ *bisects* $\overline{MJ}$ *and* $\overline{FH}$.
Prove: $\overline{MJ} \cong \overline{FH}$



Proof

Statements	Reasons
1 $\overline{MK} \cong \overline{FG}$	1 Given
2 $\overline{KG}$ bisects $\overline{MJ}$ and $\overline{FH}$.	2 Given
3 $\overline{MJ} \cong \overline{FH}$	3 If segments are $\cong$, their like multiples (doubles) are $\cong$. (Multiplication Property)

Problem 4 Given: $\angle NOP \cong \angle RPO$;
$\overrightarrow{PT}$ *bisects* $\angle RPO$.
$\overrightarrow{OS}$ *bisects* $\angle NOP$.
$\angle NSO$ *is comp. to* $\angle 1$.
$\angle RTP$ *is comp. to* $\angle 3$.
Prove: $\angle NSO \cong \angle RTP$



Proof

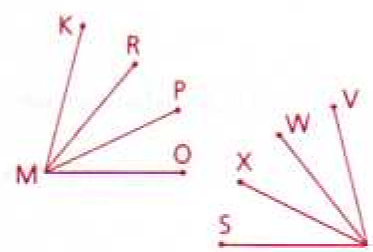
Statements	Reasons
1 $\angle NOP \cong \angle RPO$	1 Given
2 $\overrightarrow{PT}$ bisects $\angle RPO$.	2 Given
3 $\overrightarrow{OS}$ bisects $\angle NOP$.	3 Given
4 $\angle 1 \cong \angle 3$	4 Halves of $\cong$ angles are $\cong$. (An alternative form of the Division Property)
5 $\angle NSO$ is comp. to $\angle 1$.	5 Given
6 $\angle RTP$ is comp. to $\angle 3$.	6 Given
7 $\angle NSO \cong \angle RTP$	7 Complements of $\cong$ $\angle$s are $\cong$.

Part Three: Problem Sets

Problem Set A

Before starting the proofs in this problem set, reread the chart on page 90.

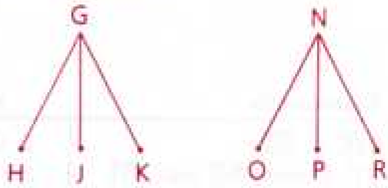
1 Given: $\angle KMR \cong \angle VTW$;
$\overrightarrow{MR}$ and $\overrightarrow{MP}$ trisect $\angle KMO$.
$\overrightarrow{TX}$ and $\overrightarrow{TW}$ trisect $\angle STV$.
Prove: $\angle KMO \cong \angle STV$



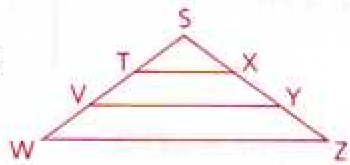
Problem Set A, *continued*

2 Use the given information to find the value of x.

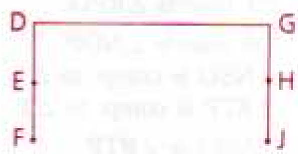
a $\angle HGJ \cong \angle ONP$;
$\overrightarrow{GJ}$ and $\overrightarrow{NP}$ are $\angle$ bisectors.
$\angle HGK = 50°$,
$\angle ONR = (2x + 10)°$



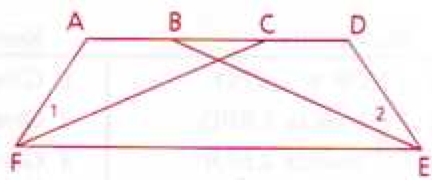
b $\overline{SW} \cong \overline{SZ}$;
$\overleftrightarrow{TX}$ and $\overleftrightarrow{VY}$ trisect $\overline{SW}$ and $\overline{SZ}$.
$ST = 12$,
$YZ = x - 4$



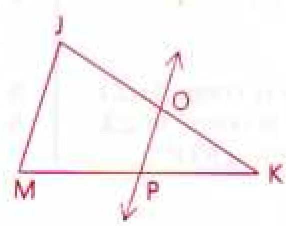
3 Given: $\overline{DF} \cong \overline{GJ}$;
E is the midpoint of $\overline{DF}$.
H is the midpoint of $\overline{GJ}$.
Prove: $\overline{DE} \cong \overline{GH}$



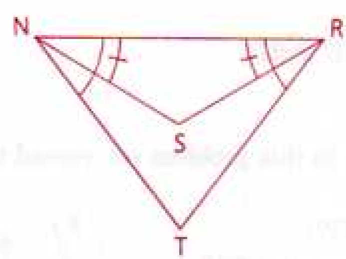
4 Given: $\angle AFE \cong \angle DEF$;
$\overrightarrow{FC}$ bisects $\angle AFE$.
$\overrightarrow{EB}$ bisects $\angle DEF$.
Conclusion: $\angle 1 \cong \angle 2$



5 Given: $\overline{JK} \cong \overline{MK}$;
$\overleftrightarrow{OP}$ bisects $\overline{JK}$ and $\overline{MK}$.
Prove: $\overline{JO} \cong \overline{PK}$

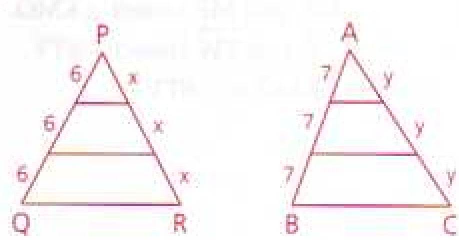


6 Given: $\angle TNR \cong \angle TRN$,
$\angle NRS \cong \angle RNS$
Conclusion: ___?___



7 **a** If $\overline{PQ} \cong \overline{PR}$ in $\triangle PQR$, what can we conclude?

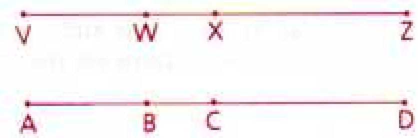
b If $AC = AB + 3$ in $\triangle ABC$, what can we conclude?



8 Given: M is the midpoint of $\overline{GH}$.
Conclusion: $\overline{GM} \cong \overline{MH}$

9 Given: $(x_1, y_1) = (5, 1)$,
$(x_2, y_2) = (9, 3)$
Find: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

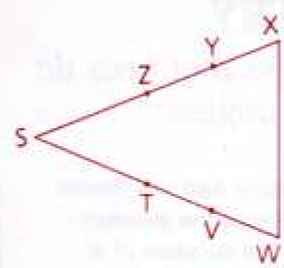
10 Copy the diagram and the proof. Then complete the proof by filling in the missing reasons.
Given: $\overline{VW} \cong \overline{AB}$, $\overline{WX} \cong \overline{BC}$;
X is the midpt. of $\overline{VZ}$.
C is the midpt. of $\overline{AD}$.
Prove: $\overline{VZ} \cong \overline{AD}$



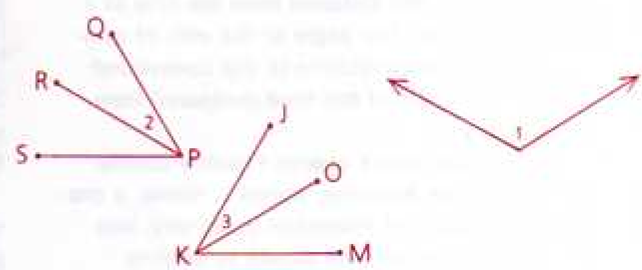
Statements	Reasons
1 $\overline{VW} \cong \overline{AB}$	1 ________
2 $\overline{WX} \cong \overline{BC}$	2 ________
3 $\overline{VX} \cong \overline{AC}$	3 ________
4 X is the midpt. of $\overline{VZ}$.	4 ________
5 C is the midpt. of $\overline{AD}$.	5 ________
6 $\overline{VZ} \cong \overline{AD}$	6 ________

Problem Set B

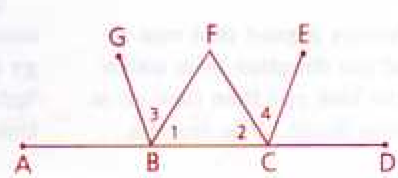
11 Given: $\overline{SZ} \cong \overline{ST}$,
$\overline{XY} \cong \overline{VW}$;
Y is the midpt. of $\overline{ZX}$.
V is the midpt. of $\overline{TW}$.
Prove: $\overline{SX} \cong \overline{SW}$



12 Given: $\overrightarrow{PR}$ bisects $\angle QPS$.
$\overrightarrow{KO}$ bisects $\angle JKM$.
$\angle 1$ is supp. to $\angle JKM$.
$\angle 1$ is supp. to $\angle QPS$.
Conclusion: $\angle 2 \cong \angle 3$



13 Given: $\angle 1 \cong \angle 2$;
$\overrightarrow{BG}$ bisects $\angle ABF$.
$\overrightarrow{CE}$ bisects $\angle FCD$.
Prove: $\angle 3 \cong \angle 4$

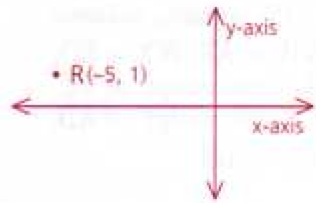


Problem Set B, *continued*

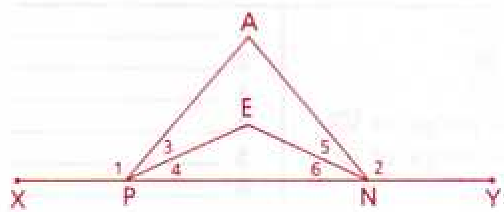
14 If four times the supplement of an angle is added to eight times the angle's complement, the sum is equivalent to three straight angles. Find the measure of the angle that is supplementary to the complement.

Problem Set C

15 Point T is located on the graph so that $\overleftrightarrow{RT}$ is perpendicular to the x-axis and $3 < RT < 5$. Find the restrictions on the coordinates of T.



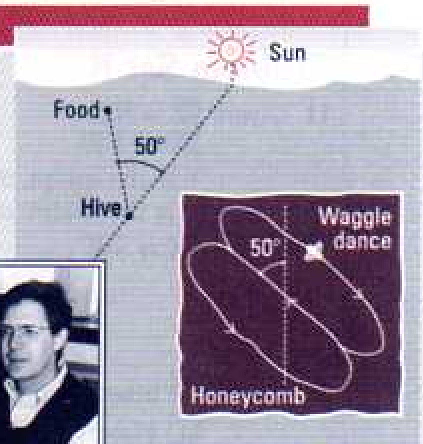
16 Given: $\angle 1 \cong \angle 2$,
$\overrightarrow{PE}$ bis. $\angle APN$,
$\overrightarrow{NE}$ bis. $\angle ANP$
Prove: $\angle XPE \cong \angle ENY$



CAREER PROFILE

BEE GEOMETRY

James L. Gould shows that bees do indeed know about angles



In the early 1900's, zoologist Karl von Frisch showed that bees convey information geometrically by "waggle dancing." The duration of a dance conveys the distance from the hive of a new food source. The *angle* of the axis of symmetry of the dance relative to the honeycomb conveys the angle of the food measured from the sun line.

Behaviorists didn't accept Frisch's conclusions. Recently, however, James L. Gould, a professor of biology at Princeton University, has conducted research that seems to confirm Frisch's results.

Opponents of the theory argued that new recruits simply observed the direction from which dancers returned to the hive and then flew off in that direction," explains Gould. "I've found a way to make dancers lie. Recruits still followed the dance directions."

Geometry comes naturally to bees. "They're wired for it," explains Gould. "It's like a computer program in their brains."

Gould has a bachelor's degree in molecular biology from the California Institute of Technology and a doctorate from Rockefeller University. Today he is a professor of biology at Princeton University.

2.7

Transitive and Substitution Properties

Objectives

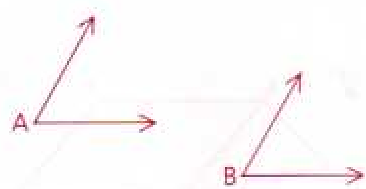
After studying this section, you will be able to

- Apply the transitive properties of angles and segments
- Apply the Substitution Property

Part One: Introduction

Transitive Properties

Suppose that ∠A ≅ ∠B and ∠A ≅ ∠C. Is ∠B ≅ ∠C?



The transitive property of algebra can be used to prove this general rule.

Theorem 16 ***If angles (or segments) are congruent to the same angle (or segment), they are congruent to each other. (Transitive Property)***

Theorem 16 can be used twice to prove the next theorem.

Theorem 17 ***If angles (or segments) are congruent to congruent angles (or segments), they are congruent to each other. (Transitive Property)***

Substitution Property

In your algebra studies and in some of the problems you have worked this year, you have solved for a variable such as x and then ***substituted*** the value you found for that variable.

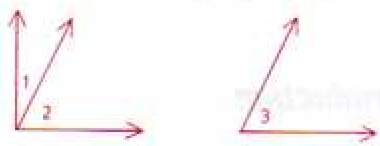
Example *If* $\angle A \cong \angle B$, *find* $m\angle A$.

$$2x - 4 = x + 10$$
$$x = 14$$

We can now substitute 14 for x in $m\angle A = x + 10$ to find that $m\angle A = 14 + 10 = 24$.

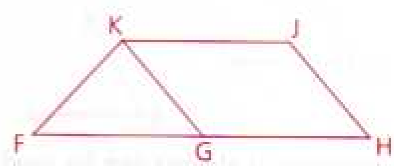
The Substitution Property can also be applied when no variables are involved.

If $\angle 1$ is comp. to $\angle 2$ and $\angle 2 \cong \angle 3$, then $\angle 1$ is comp. to $\angle 3$ by ***substitution.***



Part Two: Sample Problems

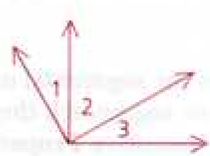
Problem 1 *Given:* $\overline{FG} \cong \overline{KJ}$, $\overline{GH} \cong \overline{KJ}$
Prove: $\overleftrightarrow{KG}$ *bisects* $\overline{FH}$.



Proof

Statements	Reasons
1 $\overline{FG} \cong \overline{KJ}$	1 Given
2 $\overline{GH} \cong \overline{KJ}$	2 Given
3 $\overline{FG} \cong \overline{GH}$	3 If segments are ≅ to the same segment, they are ≅. (Transitive Property)
4 $\overleftrightarrow{KG}$ bisects $\overline{FH}$.	4 If a line divides a segment into two ≅ segments, it bisects the segment.

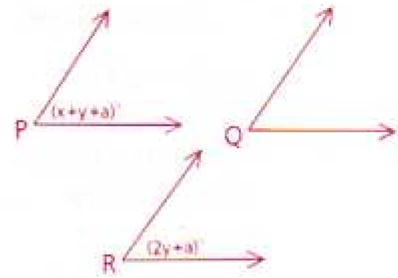
Problem 2 *Given:* $\angle 1 + \angle 2 = 90°$, $\angle 1 \cong \angle 3$
Prove: $\angle 3 + \angle 2 = 90°$



Proof

Statements	Reasons
1 $\angle 1 + \angle 2 = 90°$	1 Given
2 $\angle 1 \cong \angle 3$	2 Given
3 $\angle 3 + \angle 2 = 90°$	3 Substitution (step 2 in step 1)

Problem 3 *If ∠P ≅ ∠R and ∠Q ≅ ∠R, express m∠Q in terms of x and a.*



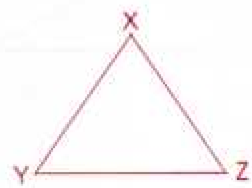
Solution

$$2y + a = x + y + a$$
$$2y = x + y$$
$$y = x$$
$$m\angle P = x + y + a = x + x + a$$
$$m\angle Q = 2x + a$$

Part Three: Problem Sets

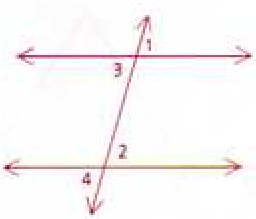
Problem Set A

1 Given: ∠X ≅ ∠Y,
∠X ≅ ∠Z
Conclusion: ∠Y ≅ ∠Z

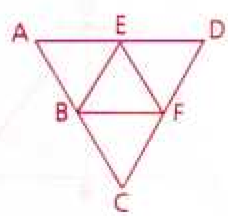


2 Given: ∠1 ≅ ∠2,
∠2 ≅ ∠3
Conclusion: ∠1 ≅ ∠3

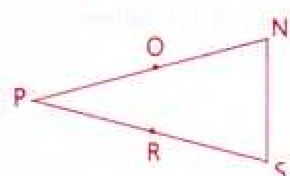
3 Given: ∠1 ≅ ∠3,
∠2 ≅ ∠3,
∠2 ≅ ∠4
Prove: ∠1 ≅ ∠4



4 Given: BC + BE = AD,
BE = EF
Prove: BC + EF = AD



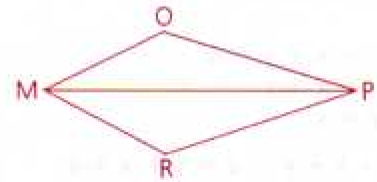
5 Given: O is the midpt. of $\overline{NP}$.
R is the midpt. of $\overline{SP}$.
$\overline{NP} \cong \overline{SP}$
Conclusion: $\overline{SR} \cong \overline{NO}$



6 Given: $\overline{GJ} \cong \overline{HK}$
Conclusion: $\overline{GH} \cong \overline{JK}$

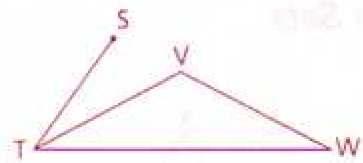
Problem Set A, *continued*

7 Given: $\angle OMP \cong \angle RPM$;
$\overrightarrow{MP}$ bisects $\angle OMR$.
$\overrightarrow{PM}$ bisects $\angle OPR$.
Prove: $\angle OMR \cong \angle OPR$



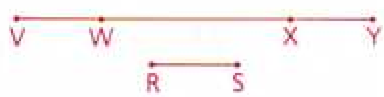
8 The complement of an angle is 24° greater than twice the angle. Find the measure of the complement.

9 $\angle W \cong \angle STV$;
$\overrightarrow{TV}$ bisects $\angle STW$.
$\angle W = (2x - 5)^\circ$,
$\angle VTW = (x + 15)^\circ$
Find: $m\angle STW$

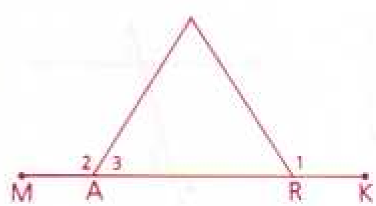


Problem Set B

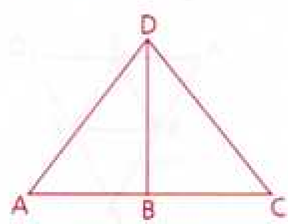
10 Given: $\overline{VW} \cong \overline{RS}$,
$\overline{XY} \cong \overline{RS}$
Prove: $\overline{VX} \cong \overline{WY}$



11 Given: $\angle 1 \cong \angle 2$
Conclusion: $\angle 1$ is supp. to $\angle 3$.



12 Given: $\angle A$ is comp. to $\angle ADB$.
$\angle C$ is comp. to $\angle CDB$.
$\overrightarrow{DB}$ bisects $\angle ADC$.
Conclusion: $\angle A \cong \angle C$

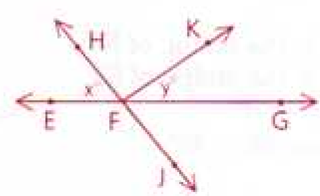


13 Find the measures of each of the following angles in terms of x and y.

a $\angle HFK$

b $\angle EFK$

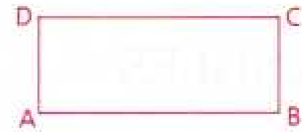
c $\angle HFG$



14 When one-half the supplement of an angle is added to the complement of the angle, the sum is 120°. Find the measure of the complement.

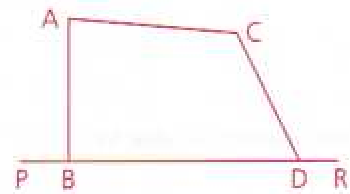
15 Given: $\angle A$ is a right $\angle$.
$\angle B$ is a right $\angle$.
$\angle B \cong \angle D$

Prove: $\angle A \cong \angle D$



Problem Set C

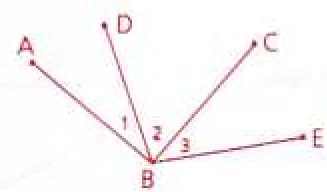
16 Given: $\overline{AB} \perp \overline{PR}$,
$\overline{AB} \cong \overline{CD}$



Fool Proof said that since $\overline{AB} \perp \overline{PR}$ and $\overline{AB} \cong \overline{CD}$, he could prove that $\overline{CD} \perp \overline{PR}$ by substitution. What is wrong with Fool's proof?

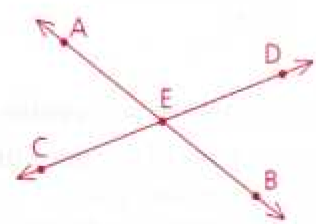
17 Given: $\overline{AB} \perp \overline{BC}$,
$\angle 1 \not\cong \angle 3$

Prove: $\angle DBE$ is a right angle.

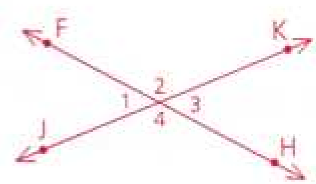


Problem Set D

18 $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E, and the ratio of $m\angle AEC$ to $m\angle AED$ is 2:3. Write an argument to show that it is impossible for $m\angle DEB$ to be 80.



19 If two of the four nonstraight angles formed by the intersection of $\overleftrightarrow{FH}$ and $\overleftrightarrow{JK}$ are selected at random, what is the probability that the two angles are congruent?



20 Find all possible values of x if x is the measure of an angle that satisfies the following set of conditions:

The angle must have a complement, and three fourths of the supplement of the angle must have a complement.

2.8 VERTICAL ANGLES

Objectives

After studying this section, you should be able to

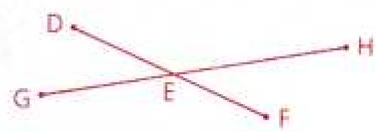
- Recognize opposite rays
- Recognize vertical angles

Part One: Introduction

Opposite Rays

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are ***opposite rays.***

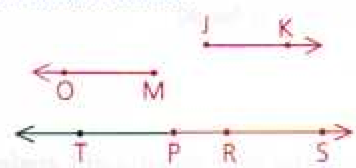
$\overrightarrow{ED}$ and $\overrightarrow{EF}$ are also opposite rays, as are $\overrightarrow{EG}$ and $\overrightarrow{EH}$.



Definition Two collinear rays that have a common endpoint and extend in different directions are called ***opposite rays***.

Some pairs of rays that are *not* opposite rays are shown below.

$\overrightarrow{JK}$ and $\overrightarrow{MO}$ are not parts of the same line.
$\overrightarrow{PT}$ and $\overrightarrow{RS}$ are not opposite, since they do not have a common endpoint.



Vertical Angles

Whenever two lines intersect, two pairs of ***vertical angles*** are formed.

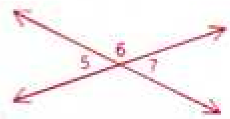
Definition Two angles are ***vertical angles*** if the rays forming the sides of one and the rays forming the sides of the other are opposite rays.

∠1 and ∠2 are vertical angles.
∠3 and ∠4 are vertical angles.

Are ∠3 and ∠2 vertical angles? How do vertical angles compare in size?

Theorem 18 ***Vertical angles are congruent.***

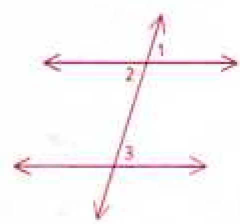
Given: Diagram as shown
Prove: ∠5 ≅ ∠7



We proved Theorem 18 in Section 2.4, sample problem 3.

Part Two: Sample Problems

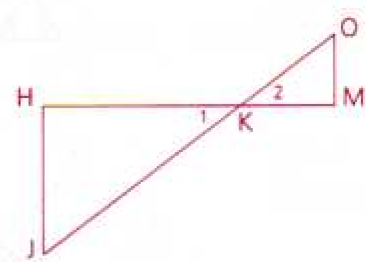
Problem 1 *Given: ∠2 ≅ ∠3*
Prove: ∠1 ≅ ∠3



Proof

Statements	Reasons
1 ∠2 ≅ ∠3	1 Given
2 ∠1 ≅ ∠2	2 Vertical angles are congruent.
3 ∠1 ≅ ∠3	3 If ∠s are ≅ to the same ∠, they are ≅. (Transitive Property)

Problem 2 *Given: ∠O is comp. to ∠2.*
∠J is comp. to ∠1.
Conclusion: ∠O ≅ ∠J

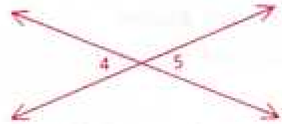


Proof

Statements	Reasons
1 ∠O is comp. to ∠2.	1 Given
2 ∠J is comp. to ∠1.	2 Given
3 ∠1 ≅ ∠2	3 Vertical angles are congruent.
4 ∠O ≅ ∠J	4 Complements of ≅ ∠s are ≅.

Problem 3 *Given:* $m\angle 4 = 2x + 5$,
$m\angle 5 = x + 30$

Find: $m\angle 4$



Solution $2x + 5 = x + 30$
$x = 25$

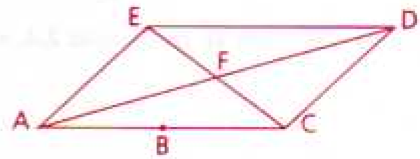
Therefore, $m\angle 4 = 2(25) + 5$, or 55.

Part Three: Problem Sets

Problem Set A

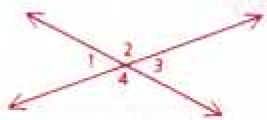
1 **a** Name three pairs of opposite rays in the diagram.

b Name two pairs of vertical angles.



2 Given: $\angle 1 = 60°32'$

Find: **a** $\angle 2$
b $\angle 3$
c $\angle 4$

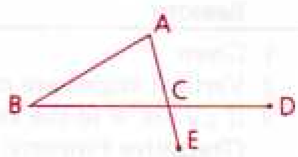


3 Given: $\angle 5 = (2x + 7)°$,
$\angle 6 = (x + 25)°$

Find: $m\angle 5$

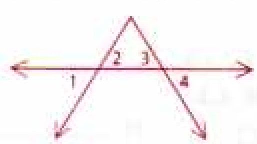
4 Given: $\angle A \cong \angle ACB$

Prove: $\angle A \cong \angle DCE$



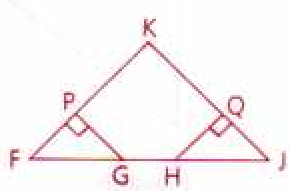
5 Given: $\angle 1 \cong \angle 4$

Conclusion: $\angle 2 \cong \angle 3$

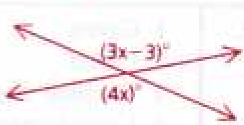


6 Given: $\overline{FH} \cong \overline{GJ}$

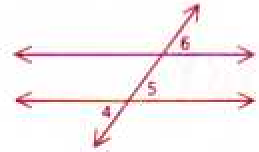
Prove: $\overline{FG} \cong \overline{HJ}$



7 Is this possible?

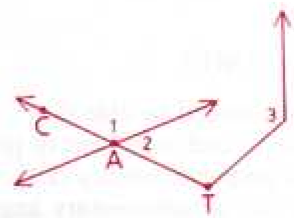


8 Given: ∠4 ≅ ∠6
Prove: ∠5 ≅ ∠6

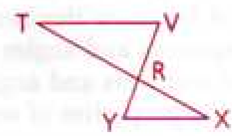


Problem Set B

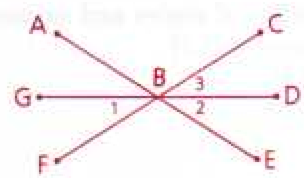
9 Given: ∠1 ≅ ∠3
Prove: ∠2 is supp. to ∠3.



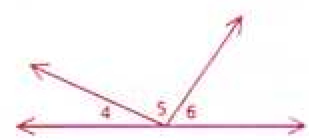
10 Given: ∠V ≅ ∠YRX,
∠Y ≅ ∠TRV
Prove: ∠V ≅ ∠Y



11 Given: $\overleftrightarrow{GD}$ bisects ∠CBE.
Conclusion: ∠1 ≅ ∠2



12 Angles 4, 5, and 6 are in the ratio 2:5:3.
Find the measure of each angle.



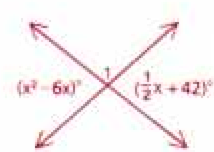
13 If a pair of vertical angles are supp., what can we conclude about the angles?

14 Graph the five points A = (3, −4), B = (0, 5), C = (0, −5), D = (−3, 4), and O = (0, 0). Which of the following are opposite rays?

a $\overrightarrow{OC}$, $\overrightarrow{OB}$ **b** $\overrightarrow{OA}$, $\overrightarrow{OD}$ **c** $\overrightarrow{BC}$, $\overrightarrow{CB}$ **d** $\overrightarrow{OB}$, $\overrightarrow{OD}$

Problem Set C

15 Find m∠1.



2

Chapter Summary

Concepts and Procedures

After studying this section, you should be able to

- Recognize the need for clarity and concision in proofs (2.1)
- Understand the concept of perpendicularity (2.1)
- Recognize complementary and supplementary angles (2.2)
- Follow a five-step procedure to draw logical conclusions (2.3)
- Prove angles congruent by means of four new theorems (2.4)
- Apply the addition properties of segments and angles (2.5)
- Apply the subtraction properties of segments and angles (2.5)
- Apply the multiplication and division properties of segments and angles (2.6)
- Apply the transitive properties of angles and segments (2.7)
- Apply the Substitution Property (2.7)
- Recognize opposite rays (2.8)
- Recognize vertical angles (2.8)

Vocabulary

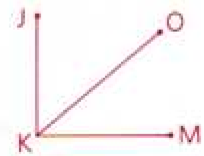
complement (2.2)
complementary angles (2.2)
coordinates (2.1)
oblique lines (2.1)
opposite rays (2.8)
origin (2.1)
perpendicular (2.1)
substitute (2.7)
substitution (2.7)
supplement (2.2)
supplementary angles (2.2)
vertical angles (2.8)
x-axis (2.1)
y-axis (2.1)

2

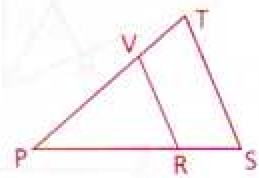
Review Problems

Problem Set A

1 Given: $\overline{JK} \perp \overline{KM}$
Prove: $\angle JKO$ is comp. to $\angle OKM$.

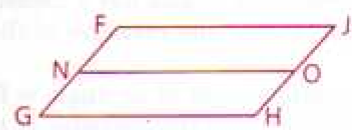


2 Given: $\overline{PV} \cong \overline{PR}$,
$\overline{VT} \cong \overline{RS}$
Conclusion: $\overline{PT} \cong \overline{PS}$

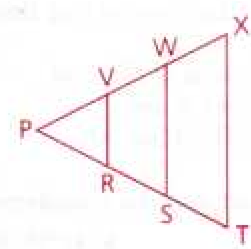


3 Given: $\angle WXT \cong \angle YXZ$
Prove: $\angle WXZ \cong \angle TXY$

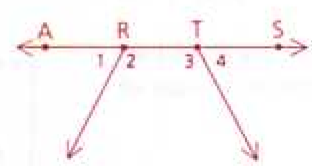
4 Given: $\overline{FG} \cong \overline{JH}$;
N is the midpt. of $\overline{FG}$.
O is the midpt. of $\overline{JH}$.
Prove: $\overline{NG} \cong \overline{OH}$



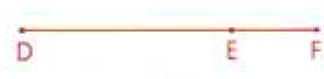
5 Given: $\overline{RV}$ and $\overline{SW}$ trisect $\overline{PT}$ and $\overline{PX}$.
$\overline{ST} \cong \overline{WX}$
Conclusion: $\overline{PT} \cong \overline{PX}$



6 Given: Diagram as shown,
$\angle 1 \cong \angle 4$
Prove: $\angle 2 \cong \angle 3$



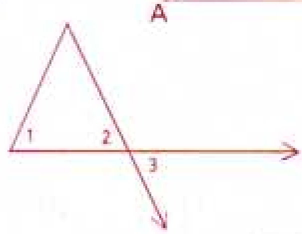
7 Point E divides $\overline{DF}$ into segments in a ratio (from left to right) of 5:2.
If DF = 21 cm, find EF.



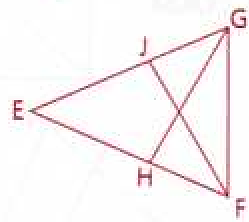
Review Problem Set A, *continued*

8 Given: ∠A is supp. to ∠D.
∠A ≅ ∠C
Prove: ∠C is supp. to ∠D.

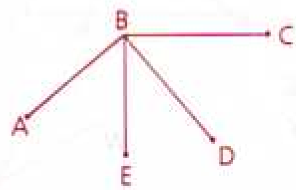
9 Given: ∠1 ≅ ∠3
Conclusion: ∠1 ≅ ∠2



10 Given: ∠EGF ≅ ∠EFG,
∠EGH ≅ ∠EFJ
Conclusion: ∠HGF ≅ ∠JFG



11 Given: ∠ABD is a right ∠.
∠CBE is a right ∠.
Conclusion: ∠ABE ≅ ∠CBD



12 One of two complementary angles has a measure that is six more than twice the other's. Find the measure of the larger angle.

13 The meaure of the supplement of an angle is five times that of the angle's complement. Find the measure of the complement.

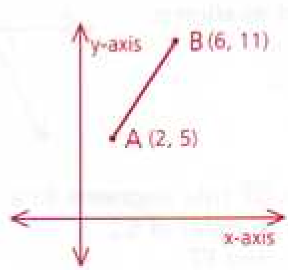
14 Two nonperpendicular intersecting lines are called __?__.

15 Point A is the midpoint of $\overline{DE}$, and DA = 12. Points I and N are trisection points of $\overline{DE}$. Find AN.

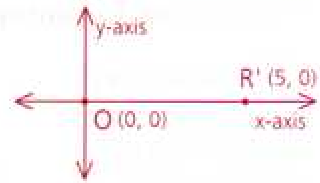
16 Find the supplement and the complement of each angle.

a 83° **b** 42°15′38″ **c** 97°

17 If $\overline{AB}$ is reflected over the x-axis, what will the coordinates of the endpoints of the reflection be?

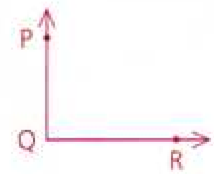


18 A point, R, was rotated about the origin, first 180° clockwise and then 90° counterclockwise. It ended at R′ = (5, 0). Find the coordinates of point R.



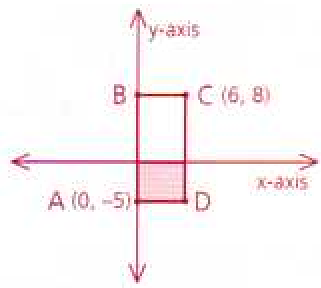
19 ∠PQR is a right angle. If $\overrightarrow{QS}$ is drawn at random between the sides of ∠PQR, what is the probability that

a ∠PQS and ∠SQR are complementary?
b ∠PQS is between 0° and 45°?



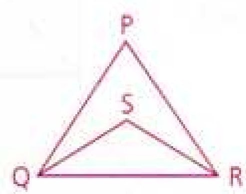
20 ABCD is a rectangle.

a Find the coordinates of B and D.
b If a point within ABCD is picked at random, what is the probability that it is in the shaded region?



21 Given: ∠PQR ≅ ∠PRQ;
$\overrightarrow{QS}$ bisects ∠PQR.
$\overrightarrow{RS}$ bisects ∠PRQ.
∠PQR = 87°26′

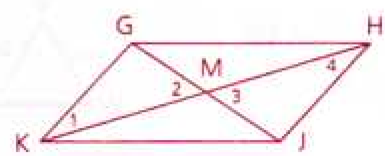
Find: ∠PRS



Problem Set B

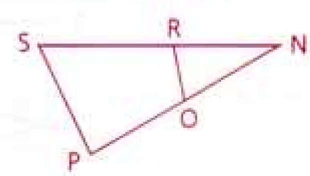
22 Given: ∠1 is comp. to ∠3.
∠4 is comp. to ∠2.

Conclusion: ∠1 ≅ ∠4



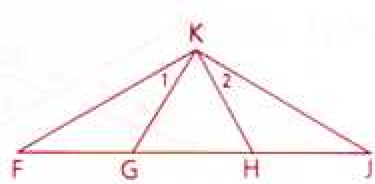
23 Given: O is the midpoint of $\overline{NP}$.
$\overline{RN} \cong \overline{PO}$

Conclusion: $\overline{RN} \cong \overline{NO}$



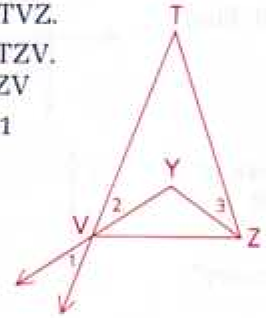
24 Given: ∠F ≅ ∠1,
∠J ≅ ∠2,
$\overline{FK} \perp \overline{KH}$,
$\overline{GK} \perp \overline{KJ}$

Prove: ∠F ≅ ∠J

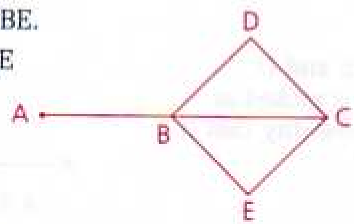


Review Problem Set B, *continued*

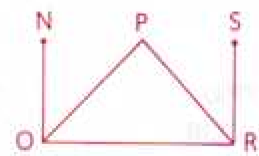
25 Given: $\overrightarrow{VY}$ bisects $\angle TVZ$.
$\overrightarrow{ZY}$ bisects $\angle TZV$.
$\angle TVZ \cong \angle TZV$
Conclusion: $\angle 3 \cong \angle 1$



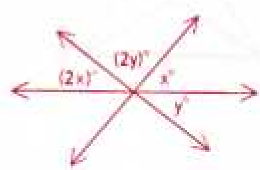
26 Given: $\overrightarrow{BC}$ bisects $\angle DBE$.
Prove: $\angle ABD \cong \angle ABE$



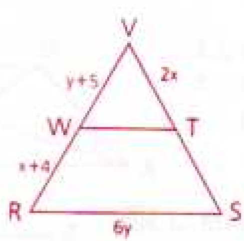
27 Given: $\angle NOP \cong \angle SRP$;
$\angle NOP$ is comp. to $\angle POR$.
$\angle SRP$ is comp. to $\angle PRO$.
Prove: $\angle POR \cong \angle PRO$



28 Solve for x and y.

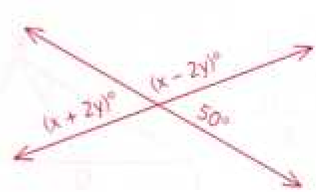


29 Given: $\overline{VS} \cong \overline{VR}$;
$\overline{WT}$ bisects $\overline{VS}$ and $\overline{VR}$.
Find: The perimeter of $\triangle VRS$



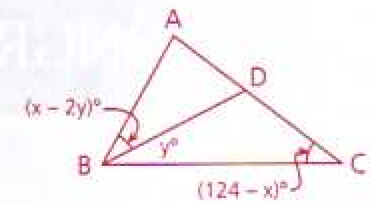
30 Solve for y in terms of x

31 By how much does x exceed y?



32 The measure of the supplement of an angle exceeds twice the measure of the complement of the angle by 20. Find the measure of half of the complement.

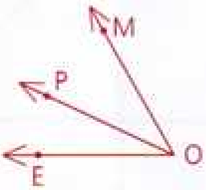
33 $\overrightarrow{BD}$ bisects ∠ABC.



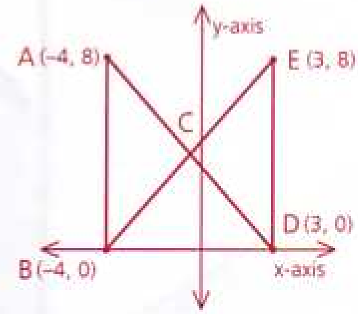
a Write an equation that relates x and y.
b If ∠DBC ≅ ∠C, write another equation relating x and y.
c Use substitution with parts **a** and **b** to find m∠C.

34 Given: $\overrightarrow{OP}$ bisects ∠MOE.
m∠MOP = 10 − 3x,
m∠POE = $x^2 - 6x$

Find: m∠MOE



35 **a** Find the area of △BDE.
b How does the area of △ABC compare with the area of △EDC?



Problem Set C

36 With respect to the origin, point A = (1, 2) is rotated 100° clockwise, then 80° counterclockwise, then 210° clockwise, and finally 50° counterclockwise to point B.

a Find the coordinates of point B.
b After which of the four rotations was the point in Quadrant I?

37 Tippy Van Winkle is awakened from a deep sleep by the cuckoo of a clock that sounds every half hour. Before Tippy can look at the clock, his brother Bippy enters the room and offers to bet $10 that the hands of the clock form an acute angle. Assuming that the hands have not moved since the cuckoo sounded, how much should Tippy put up against Bippy's $10 so that it is an even bet?

38 Given: ∠ABD is supp. to ∠EDB.
$\overrightarrow{BC}$ bisects ∠ABD.
$\overrightarrow{DC}$ bisects ∠BDE.

Prove: ∠CBD is comp. to ∠BDC.
(Use a paragraph proof.)