Notes 7.5- Operations with Radical Expressions

**Why?**
Golden rectangles have been used by artists and architects to create beautiful designs. Many golden rectangles appear in the Parthenon in Athens, Greece. The ratio of the lengths of the sides of a golden rectangle is $\frac{2}{\sqrt{5} - 1}$. In this lesson, you will learn to simplify radical expressions like $\frac{2}{\sqrt{5} - 1}$.

**Simplify Radicals:** The properties you have used to simplify radical expressions involving square roots also hold true expressions involving $n$th roots. (page 439).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$n^4$</th>
<th>$n^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>3</td>
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<td>5</td>
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**Key Concept - Product Property of Radicals**

Words: For any real numbers $a$ and $b$ and any integer $n > 1$,

\[ \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}, \text{ if } n \text{ is even and } a \text{ and } b \text{ are both nonnegative or if } n \text{ is odd.} \]

Examples: $\sqrt{2} \cdot \sqrt{8} = \sqrt{16} \text{ or } 4$ and $\sqrt{3} \cdot \sqrt{9} = \sqrt{27} \text{ or } 3$.

In order for a radical to be insimplest form, the radical must contain no factors that are $n$th powers of an integer or polynomial.
Directions: Simplify

a. $\sqrt{25a^4b^9}$  

b. $3\sqrt[3]{125m^{30}p^{20}}$  
c. $3\sqrt[3]{-108x^9y^{12}z^{17}}$

The quotient Property of Radicals is another property used to simplify radicals.

Key Concept

Quotient Property of Radicals

| Words | For any real numbers $a$ and $b \neq 0$ and any integer $n > 1$, 
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, if all roots are defined. |
|-------|---------------------------------------------------------------------|
| Examples | $\frac{\sqrt{27}}{\sqrt{3}} = \sqrt{9}$ or $3$  
$\sqrt[3]{\frac{x^6}{8}} = \frac{\sqrt[3]{x^6}}{\sqrt[3]{8}} = \frac{x^2}{2}$ or $\frac{1}{2}x^2$ |

To eliminate radicals from a denominator or fractions from a radicand, you can use a process called **rationalizing the denominator**. To rationalize, multiply the numerator and denominator by quantity so that the radicand has an exact root.

<table>
<thead>
<tr>
<th>If the denominator is:</th>
<th>Multiply the numerator &amp; denominator by:</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{b}$</td>
<td>$\sqrt{b}$</td>
<td>$\frac{2}{\sqrt{5}} =$</td>
</tr>
<tr>
<td>$\sqrt[n]{b^x}$</td>
<td>$\sqrt[n]{b^{n-x}}$</td>
<td>$\frac{3}{\sqrt[4]{2}} =$</td>
</tr>
</tbody>
</table>
Directions: Simplify

\[ \sqrt{\frac{y^8}{x^7}} \quad \text{b.} \quad \sqrt[3]{\frac{2}{9x}} \quad \text{c.} \quad \sqrt[3]{\frac{8}{25}} \]

Here is a summary of the rules used to simplify radicals:

**Concept Summary: Simplifying Radical Expressions**

A radical expression is in simplified form when the following conditions are met:

- The index \( n \) is as small as possible.
- The radicand contains no factors (other than 1) that are \( n \)th powers of an integer or polynomial.
- The radicand contains no fractions.
- No radicals appear in a denominator.

**Operations with Radicals:** You can use the Product and Quotient Properties to multiply and divide some radicals.

Directions: Simplify

\[ a. \quad 5^3\sqrt{100a^2} \cdot 3^3\sqrt{10a} \quad \text{b.} \quad 2^4\sqrt{8x^3y^2} \cdot 3^4\sqrt{2x^5y^2} \]
Radicals can be added and subtracted if the radicals are like terms. Radicals are **like radical expressions** if both the index and the radicand are identical.

**EXAMPLE 4** Add and Subtract Radicals

**Directions: Simplify**

a. \(3\sqrt{45} - 5\sqrt{80} + 4\sqrt{20}\)  
b. \(2\sqrt[3]{125a^4} - 5\sqrt[3]{8a}\)

You can also multiply radicals using the FOIL method as you do when multiplying binomials.

**EXAMPLE 5** Multiply Radicals

**Directions: Simplify**

a. \((2\sqrt{3} + 3\sqrt{5})(3 - \sqrt{3})\)  
b. \((7\sqrt{2} - 3\sqrt{3})(7\sqrt{2} + 3\sqrt{3})\)

Binomials of the form \(a\sqrt{b} + c\sqrt{d}\) and \(a\sqrt{b} - c\sqrt{d}\), where \(a, b, c,\) and \(d\) are rational numbers, are called **conjugates** of each other. You can use conjugates to rationalize denominators.
**ARCHITECTURE:** Refer to the beginning of the lesson and simplify the expression \( \frac{2}{\sqrt{5} - 1} \).

\[
\frac{6 - \sqrt{3}}{\sqrt{3} + 4}
\]

**Notes 7.6- Rational Exponents**

**Why?**

The formula \( C = c(1 + r)^n \) can be used to estimate the future cost of an item due to inflation. For example,

\[
C = c(1 + r)^{\frac{1}{2}}
\]

can be used to estimate the cost of a video game system in six months.

**Rational Exponents and Radicals:**

For any real number \( b \) and any positive integer \( n \), \( b^{\frac{1}{n}} = \sqrt[n]{b} \), except when \( b < 0 \) and \( n \) is even. When \( b < 0 \) and \( n \) is even, a complex root may exist.

**Examples:**

\[
27^{\frac{1}{3}} = \sqrt[3]{27} = 3 \quad (-16)^{\frac{1}{3}} = \sqrt[3]{-16} = 4i
\]
Rational Exponents

Let $b^\frac{1}{n}$ be an nth root of b, and let m be a positive integer.

- $b^\frac{1}{n} = \sqrt[n]{b}$
- $b^\frac{m}{n} = \left(\sqrt[n]{b}\right)^m$
- $b^{-\frac{m}{n}} = \frac{1}{b^\frac{m}{n}} = \frac{1}{\left(\sqrt[n]{b}\right)^m}, b \neq 0$

The rules for negative exponents also apply to negative rational exponents.

**EXAMPLE 1**

Radical and Exponential Forms

a. Write $a^\frac{1}{7}$ in radical form.  
   b. Write $a^\frac{7}{4}$ in radical form.

c. Write $\sqrt[w]{w}$ in exponential form.

d. Write $\sqrt[3]{c^{-5}}$ in exponential form.

**EXAMPLE 2**

Evaluate Expressions with Rational Exponents

Directions: Evaluate each expression.

a. $49^\frac{-1}{2}$  
   b. $32^\frac{2}{5}$  
   c. $64^\frac{-2}{3}$  
   d. $(-243)^\frac{-3}{5}$  
   e. $\sqrt[3]{10^{-9}}$
WEIGHTLIFTING: The formula \( M = 512 - 146,230B^{\frac{8}{5}} \) can be used to estimate the maximum total mass that a weightlifter of mass \( B \) kilograms can lift using the snatch and the clean and jerk. According to the formula, what is the maximum that a weightlifter weighing 168 kilograms can lift?

Simplify Expressions: All of the properties of powers you learned in Lesson 6-1 apply to rational exponents. Write each expression with all positive exponents. Also, any exponents in the denominator of a fraction must be positive integers. So, it may be necessary to rationalize the denominator.

<table>
<thead>
<tr>
<th>Property</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product of Power Property</td>
<td>( a^m \cdot a^n = a^{m+n} )</td>
</tr>
<tr>
<td>Power of a Power Property</td>
<td>( (a^m)^n = a^{mn} )</td>
</tr>
<tr>
<td>Power of a Product Property</td>
<td>( (ab)^m = a^m b^m )</td>
</tr>
<tr>
<td>Negative Exponent Property</td>
<td>( a^{-m} = \frac{1}{a^m}, a \neq 0 )</td>
</tr>
<tr>
<td>Quotient of Powers Property</td>
<td>( \frac{a^m}{a^n} = a^{m-n}, a \neq 0 )</td>
</tr>
<tr>
<td>Power of a Quotient Property</td>
<td>( (\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0 )</td>
</tr>
</tbody>
</table>
Directions: Simplify each expression.

\[ \frac{1}{p^4} \cdot p^4 \quad \text{b. } x^{\frac{-2}{3}} \quad \text{c. } \frac{b^3}{c^2} \cdot \frac{c}{b^3} \]

\[ \sqrt[3]{16x^4} \quad \left(27x^5y^4\right)^{\frac{1}{2}} \quad \sqrt[3]{-64x^9y^{12}z^{17}} \quad \frac{y^{\frac{1}{2}} + 1}{y^{\frac{1}{2}} - 1} \]

**Concept Summary**

**Expressions with Rational Exponents**

An expression with rational exponents is simplified when all of the following conditions are met.

- It has no negative exponents.
- It has no exponents that are not positive integers in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

**CW/HW:** Page 465# 41-65 all and page 467 #19-31 odd